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The Flexible Three Parameter Utility Function

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This brief paper introduces a flexible three parameter utility function, the FTP, which has a reasonably simple mathematical expression. It can be seen as a generalisation of the PRA utility function of Xie (2000), but it is more flexible. It encompasses other systems of utility functions including the HARA family and it can incorporate properties such as subsistence and saturation.

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1. INTRODUCTION

Many mathematical forms of utility functions have been considered in the economics and finance literature, although, as Xie (2000) has said, by far the most widely employed are those displaying constant relative risk aversion (CRRA). Xie pointed out the dangers implicit in an inappropriate assumption of CRRA and the desirability of more flexible utility functions permitting a greater range of risk aversion properties. He argued that the power risk aversion (PRA) utility function.[1]

$$u(x) = \frac{1}{\gamma}\left\{1 - \exp\left[-\gamma\left(\frac{x^{1-\sigma} - 1}{1-\sigma}\right)\right]\right\}, \quad \sigma \geq 0, \gamma \geq 0, \tag{1}$$

where x denotes income or wealth, was remarkably flexible as regards its capacity to represent constant or decreasing absolute risk aversion (CARA and DARA) as well as increasing, constant or decreasing relative risk aversion (IRRA, CRRA and DRRA).

[1]A closely related function to (1) has appeared previously in the agricultural economics literature, commencing with Saha (1993).

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While Xie's assessment is certainly valid, it is quite feasible to find a still more flexible function, which also has a reasonably simple mathematical expression, which can represent an even wider range of risk aversion behaviour. While it can be seen as a natural generalisation of the PRA function, it can incorporate properties, such as subsistence and saturation, that (1) can not. It also encompasses other systems of utility functions including the hyperbolic absolute risk aversion (HARA) family. I call this new function the flexible three parameter (FTP) utility function and its form and properties constitute the content of this short paper.

2. THE FLEXIBLE THREE PARAMETER (FTP) UTILITY FUNCTION

The flexible three parameter (FTP) utility function is

$$u(x) = \frac{1}{\gamma}\left\{1 - \left[1 - k\gamma\left(\frac{x^{1-\sigma}-1}{1-\sigma}\right)\right]^{\frac{1}{k}}\right\}. \tag{2}$$

The special case $k = 0$ is worth immediate attention since it gives the PRA. Remembering

$$\lim_{n\to\infty}\left(1 - \frac{y}{n}\right)^n = e^{-y}$$

shows that (2) then becomes (1). The permissible ranges of parameters will be discussed later, but it should be immediately noted that to avoid imaginary numbers for non integer values of $1/k$,

$$1 - k\gamma\left(\frac{x^{1-\sigma}-1}{1-\sigma}\right) > 0. \tag{3}$$

As will be seen, for some combinations of parameter values this will imply that x is bounded either below or above. This will not be a limitation on the model; rather, it will be shown to permit the representation of subsistence or saturation levels of income or wealth.

The derivative of (2) is

$$u'(x) = x^{-\sigma}\left[1 - k\gamma\left(\frac{x^{1-\sigma}-1}{1-\sigma}\right)\right]^{\frac{1}{k}-1},$$

which is real if (3) holds and is positive as required for a utility function. The second derivative is

$$u''(x) = -x^{-\sigma}\left[1 - k\gamma\left(\frac{x^{1-\sigma}-1}{1-\sigma}\right)\right]^{\frac{1}{k}-1}\left[\frac{(1-k)\gamma x^{-\sigma}}{1 - k\gamma\left(\frac{x^{1-\sigma}-1}{1-\sigma}\right)} + \frac{\sigma}{x}\right], \tag{4}$$

and must be negative if the utility function is to represent risk aversion. This will certainly be the case if we impose the additional conditions of $k \leq 1$ and γ and σ positive, but can also be true for some x outside these parameter ranges. For example, if σ is negative (4) is still negative provided

$$x^{|\sigma|+1} > \frac{|\sigma|(1 + |\sigma| + k\gamma)}{\gamma(1 + |\sigma| - k)}. \tag{5}$$

However, unless explicitly stated otherwise, γ and σ will be taken as positive in subsequent developments, although k may be either positive or negative. Note too that although the utility functions (1) and (2) are zero for $x = 1$ and negative for $x < 1$, $u'(x)$ and $u''(x)$ are still positive and negative respectively, so that the functions retain validity for $0 < x < 1$.

The Arrow-Pratt coefficient of absolute risk aversion $-u''(x)/u'(x)$ is

$$R_A = \frac{(1-k)\gamma x^{-\sigma}}{1 - k\gamma\left(\frac{x^{1-\sigma}-1}{1-\sigma}\right)} + \frac{\sigma}{x} \tag{6}$$

and the coefficient of relative risk aversion is $R_R = xR_A$. As will become evident, the FTP can cater for IARA, CARA, DARA as well as IRRA, CRRA and DRRA.

3. ENCOMPASSED TWO PARAMETER UTILITY FAMILIES

As already mentioned, the case $k = 0$ gives the PRA system, which includes some of the frequently employed elementary utility functions such as logarithmic or power utilities that give CRRA (and DARA) and the negative exponential utility that gives CARA (and IRRA). In addition, of course, the PRA can represent IRRA (if $\sigma < 1$) or DRRA (if $\sigma > 1$). Since Xie (2000) demonstrated these properties, there is little point lingering with the PRA except to note that, with $k = 0$, (5) shows that negative σ is feasible provided

$$x^{|\sigma|+1} > \frac{|\sigma|}{\gamma}$$

and (6) becomes

$$R_A = \gamma x^{|\sigma|} - \frac{|\sigma|}{x},$$

which increases with x. So the PRA could exhibit IARA under these circumstances. However, x might need to be very large if γ is small and we will see that IARA can be more generally represented with non-zero k.

The HARA family is characterised by the property that its risk tolerance (the inverse of the Arrow-Pratt coefficient of absolute risk aversion) is a linear function of wealth. This can lead to convenient simplifications in various applications, for example, linearity of the savings-consumption relationship in some economic growth models. Clearly the inverse of (6) is a linear function of x if either $\sigma = 1$ and $k = 0$, or $\sigma = 0$. The former case again leads to PRA utilities with the CRRA property, which are common to both the PRA and HARA families. The latter case covers the remaining HARA utilities that are not PRA utilities[2]. The coefficient of absolute risk aversion is

$$R_A = \frac{(1-k)\gamma}{1 + k\gamma - k\gamma x}.$$

If k is positive the denominator decreases as x increases, implying IARA. Clearly, $x < 1 + 1/(k\gamma)$ is required and in fact this is the upper bound for x obtained by putting $\sigma = 0$ in (3). The corresponding upper bound on the utility is $1/\gamma$, the same as that for the PRA utility as $x \to \infty$. This bound for x can be interpreted as a saturation level where R_A goes to infinity. For the case $k = 1/2$ it is easily seen that (2) becomes

$$u(x) = -(1 + \frac{\gamma}{4}) + (1 + \frac{\gamma}{4})x - \frac{\gamma}{4}x^2,$$

the familiar quadratic utility function. Obviously, any other value of k between zero and one will also produce a utility function with a saturation level. Taking $k = 1$ would give the risk neutral linear form.

If k is negative R_A may be written

$$R_A = \frac{(1+|k|)\gamma}{|k|\gamma x + (1 - |k|\gamma)}$$

where $x > 1 - 1/(|k|\gamma)$ and this lower bound can be interpreted as a subsistence level of income or wealth. This branch of the HARA family is commonly called the Stone-Geary class by analogy with the subsistence quantities incorporated in the famous linear expenditure commodity demand system. Since the denominator of R_A increases with x, DARA holds as x increases from the subsistence level. The branch can be further classified in terms of relative risk aversion behaviour. Since

$$R_R = \frac{(1+|k|)\gamma}{|k|\gamma + (1 - |k|\gamma)/x}$$

the denominator decreases as x increases if $|k|\gamma < 1$, so that IRRA results. If $|k|\gamma > 1$ the denominator increases as x increases, so DRRA results.

[2]However, the special case $k = 0$ and $\sigma = 0$, which gives CARA, is also a PRA case.

Many other two parameter special cases of the FTP are possible. For example, taking $\sigma = 1$ in (2) gives

$$u(x) = \frac{1}{\gamma}\left[1 - (1 - k\gamma \log x)^{\frac{1}{k}}\right],$$

where, from (3),

$$x < e^{\frac{1}{k\gamma}}$$

if k is positive. For small γ this need not restrict the range of x greatly. For this case (6) gives

$$R_A = \frac{(1-k)\gamma}{x(1 - k\gamma \log x)} + \frac{1}{x}$$

and since there is an upper bound (for positive k) IARA must eventually hold as x increases. But initially decreases as x increases[3] so that DARA will hold through some interval at first. Since

$$R_R = \frac{(1-k)\gamma}{1 - k\gamma \log x} + 1$$

and the denominator decreases as x increases, IRRA holds over the full range of x. For k negative it is easily verified that both DARA and DRRA hold. So the relative risk aversion properties of (7) are similar to those of (1) with the sign of k playing the same role as that of $1 - \sigma$.

However, the evolving absolute risk aversion behaviour exhibited by (7) for positive k might sometimes be a relevant property.

Another two-parameter utility is obtained by setting $k = 1 - \sigma$ in (2) giving

$$u(x) = \frac{1}{\gamma}\left\{1 - [1 - \gamma(x^{1-\sigma} - 1)]^{\frac{1}{1-\sigma}}\right\}.$$

This has a saturation level of income and IRRA if $\sigma < 1$, a subsistence level and DRRA if $\sigma > 1$, and some potential for non-monotonic absolute risk aversion properties.

4. THE THREE PARAMETER UTILITY AGAIN

The previous section comprised special two parameter cases of (2). For the full three parameter form, the situations of lower and upper bounds

[3]The derivative of R_A evaluated at $x = 1$ is easily seen to be $k(1-k)\gamma^2 - (1-k)\gamma - 1$, which must be negative unless γ is large.

corresponding to subsistence and saturation levels can again occur. But the unbounded situation can also arise without having to set $k = 0$. For $k > 0$, (3) implies that

$$x^{1-\sigma} < 1 + \frac{1-\sigma}{k\gamma}.$$

For $\sigma < 1$, this is a finite upper bound as long as neither k nor γ go to zero. But for $\sigma > 1$,

$$x^{1-\sigma} > 1 - \frac{\sigma - 1}{k\gamma}$$

and if $(\sigma - 1)/(k\gamma) \geq 1$, this places no restriction on x other than that it be positive, as it certainly is. If k is negative and $\sigma < 1$

$$x^{1-\sigma} > 1 - \frac{1-\sigma}{|k|\gamma},$$

which again places no bound on x if $(\sigma - 1)/(|k|\gamma) \geq 1$ and otherwise gives a lower bound. For k negative and $\sigma > 1$

$$x^{\sigma-1} > \left(1 + \frac{\sigma - 1}{|k|\gamma}\right)^{-1},$$

giving a lower bound for x between 0 and 1.

Not surprisingly, the absolute risk aversion properties depend on functions of parameters closely related to those determining bounds. The R_A formula (6) may be rewritten

$$R_A = \frac{1}{x}\left[\frac{(1-k)\gamma}{x^{\sigma-1}\left(1 + \frac{k\gamma}{1-\sigma}\right) - \frac{k\gamma}{1-\sigma}} + \sigma\right].$$

The first term in the product, the reciprocal of x, always decreases with x. The behaviour of the second term depends on the sign and size of k and size of σ. For $\sigma > 1$ and $k\gamma/(1-\sigma) > -1$ this term also decreases so DARA holds for all x. But IARA can hold at other parameter values and if there is a saturation level IARA must hold as it is approached, even if DARA holds at lower x.

Relative risk aversion properties are more easily summarised. It is easy to show that

$$\frac{\partial R_R}{\partial x} = \frac{(1-k)\gamma x^{\sigma-2}(1 + k\gamma - \sigma)}{\left[x^{\sigma-1}\left(1 + \frac{k\gamma}{1-\sigma}\right) - \frac{k\gamma}{1-\sigma}\right]^2}.$$

So IRRA holds for all x if $i + k\gamma > \sigma$ and DRRA holds for all x if $i + k\gamma < \sigma$.

5. CONCLUDING REMARKS

The FTP utility function has been shown to encompass not only the PRA function, but also other forms including utilities embodying lower or upper bounds on x. Is this really a useful property? The considerable employment in the economic literature of forms of HARA implying lower or upper bounds, suggests that in at least some applied problems the concepts of subsistence or saturation levels are considered useful. The FTP caters for them in more general forms than does HARA. Perhaps more importantly, estimating the three parameters of the utility function from actual data will fit the model most compatible with the data and permit tests of the reality of hypothesised bounds through observing whether the inequalities of the previous section are satisfied or not. It is beyond the scope of this paper to discuss the actual econometrics of estimation and testing. However, there are well known approaches for other situations that seem econometrically quite similar. A particularly relevant case is that of estimating the three parameters of the generalised extreme value distribution[4](GEV). One parameter plays a corresponding role to k in (2). If it is zero the distribution reduces to a two parameter distribution, the Gumbel, without bounds on x. If it is positive the three parameter distribution is reversed Weibull with an upper bound on x and if negative it is Frechet with a lower bound[5].

As mentioned in the previous section, x can be unbounded without k necessarily being zero in the FPT. So even in cases where there might be a priori knowledge that subsistence or saturation levels do not exist, the extra parameter in the FTP does give it some greater flexibility than two parameter rivals.

Ultimately, of course, the value of a new utility function is best assessed by how useful and informative it proves to be in empirical studies. Perhaps this short note will stimulate such investigation.

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[4]See, for example, Coles (2001).

[5]Indeed the distribution function of the GEV (with location parameter set to zero) can itself provide a valid utility function, but it is much less flexible than the FTP.