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The Optimal Price of Default*

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With assets taken to be pools, rational expectations on their delivery rates are, as default is permissible in an economy and its penalty prescribed in terms of utility, indispensable to the existence of equilibrium. And the resulting equilibrium relies heavily on the prevailing penalty level. We propose, in this paper, a path-following algorithm for calculating the level of penalty that leads to a Pareto efficient equilibrium—Pareto efficient among the set of equilibria of the economies with distinct penalty levels. This algorithm brings off those rational expectations by identifying their upper and lower bounds iteratively until the discrepancies between them are admissible.

Key Words: Default; Penalty; Path-following; Pareto efficiency; Rational expectation.

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1. INTRODUCTION

The purpose of this paper is to reckon the level of penalty that yields an equilibrium of Pareto efficiency when default is permitted. Various models involving default have been proposed in the literature. The one

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used here, as in Zame (1993), is that presented in Dubey et al. (2005), where perfect competition is postulated, assets are thought of as pools, and penalty is prescribed in terms of utility.[1] With default, the equilibrium of an economy, under standard assumptions, still exists, provided households in it possess rational expectations on delivery rates of assets (see Dubey et al. (2005)). And to default, penalty pertains, whose level significantly matters; to distinct levels, a household would adapt himself by altering his behavior, which will, in turn, occasion different equilibria of an economy, one possibly Pareto dominating another. The focus of this paper is thus on seeking out a Pareto efficient equilibrium.

When the market is complete, a (Walrasian) equilibrium, as is well-known, amounts to Pareto efficiency. But seldom has an equilibrium been Pareto efficient, or even constrained Pareto efficient, when the market structure is incomplete.[2] So by a Pareto efficient equilibrium here is meant that the equilibrium is Pareto efficient in the very set of equilibria resulting from different levels of penalty. A penalty level leading to a Pareto efficient equilibrium will in the sequel be referred to as an optimal one (see also section 3.2).

In effect, default parallels assets in facilitating risk-sharing among households. By buying and selling assets, the marginal utility of money is equalized in each state when the market is complete. But with an incomplete set of markets, this may not be achievable. Allowing for default can, in certain cases, reduce the gap between them; improvement in welfare may accrue thereby.[3] Such improvement, of course, hinges upon the penalty level of default (which can be deemed its price); too severe or too light a penalty would necessarily attenuate the role of default in improving efficiency. In

[1]For other models, one is referred to Diamond (1984), Gale and Hellwig (1985), Geanakoplos (2002), and Hart and Moore (1998), etc. In the model of Geanakoplos (2002), the seller of an asset is, in case he defaults, obliged to put up collateral—another characterization of an asset in addition to its payoff (or promise). Two assets identical in payoff may differ in the quantity of collateral required. Since only scare resources can act as collateral, such an obligation confers an endogenous asset structure on the model, as is also the case for Dubey et al. (2005).

[2]For some examples on optimality of incomplete markets, see Borch (1962), Diamond (1967), Geanakoplos (1986), Hart (1975), Levine and Zame (2002), Stiglitz (1982), and Theorem 11.6 of Magill and Quinzii (1996). Borch (1962) established that a complete market structure is necessary for an equilibrium to be Pareto efficient, and this is illustrated by a two-period pure exchange economy in Hart (1975). Theorem 11.6 of Magill and Quinzii (1996) shows that for *almost* all endowment structure, Pareto efficiency of an equilibrium can not be achieved when the market is incomplete. In pursuit of a weaker sense of optimality, constrained Pareto efficiency is introduced in Diamond (1967); but even this suboptimality is out of reach when more than one commodity is available in an economy (see Stiglitz (1982)).

[3]This point is exemplified in Dubey et al. (2000, 2005) and Zame (1993). In Zame (1993), the superiority of allowing for default over opening new markets is also explored.

the present paper, we are, therefore, striving to compute the magnitudes of penalty that result in the least extent of this attenuation.

One prevalent option for such computation is an algorithm of path-following type.[4] Yet, of adopting such an algorithm, two questions put obstacles in the way: Specify a starting point and render differentiable the demand function. For the former, we modify the payoff matrix of the assets and the endowment structure of the households, so that no economic activity would, at a pretty high level of penalty, take place at equilibrium; and this trivial equilibrium would then serve well as a starting point.

For the latter, non-differentiability of the demand function may arise out of boundary conditions, inequality constraints, as well as non-strict concavity of the utility function. For boundary conditions and inequality constraints, we borrow the idea of interior-point methods from optimization, which not only helps effect a strictly concave utility function, but enjoys a fine economic interpretation. Through this idea, a demand function that is differentiable will be attained.

To acquire the optimal penalty level, still another question—how to carve out rational expectations on delivery rates of assets—remains to be settled. Rational expectation, roughly, requires one to expect neither too high nor too low a delivery rate. When the expectation is relatively high, it is to be curtailed by the upper bounds, to be determined in this paper, of rational expectations; and to be uplifted by lower bounds when it is relatively low. We, then, reduce the upper bounds and boost the lower bounds repeatedly, until the rational expectations come along.

As stated above, the magnitude of penalty figures prominently. Observe that an asset in the real economy is not characterized only by its payoff. One may cherish pension more than insurance, and insurance more than bond .... Whoever defaults on the pension is to incur a higher penalty than defaults on the others. On this score, the penalty level of each asset is supposed to be a multiple of a single (yet unknown) number, pension enjoying a higher multiplier than insurance and insurance higher than bond. In this paper, the penalty levels of assets are assumed to form an arithmetic series.

The rest of this paper is organized as follows. Section 2 describes the economy under study. Section 3 presents the algorithm for computing the optimal level of penalty, and submits this algorithm to test. Section 4 concludes the paper with a few remarks.

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[4] For an elaboration of the path-following algorithm, one is referred to, among others, Allgower and Georg (1993), Eaves (1972), Eaves and Schmedders (1999), and Scarf (1973); for its application, to Brown et al. (1996), where a path-following algorithm is designed to compute equilibria in the GEI case; for its variant, to Kellogg et al. (1976) and Smale (1976).

2. THE ECONOMY

We study here a two-period economy—call them period 0 and 1— with one commodity only.[5] Assume that there are J assets, $\hbar$ households, and S states in period 1. Name period 0 state 0, and index by s the commodity in state s. Denote the payoff matrix of the assets by $R \in \mathscr{R}_+^{S\times J}$, which specifies quantity of the commodity to be delivered by each asset in each state. Let $\mathbb{S} = \{1, \cdots, S\}$, $\mathbb{H} = \{1, \cdots, \hbar\}$, $\mathbb{J} = \{1, \cdots, J\}$; let $\mathbf{0}$ and $\mathbf{1}$ be matrices consisting respectively of zeros and ones only, their dimensions determined by the context; and let all vectors in this paper be column ones.

2.1. Characterization of a household

A household solves

$$\{\max\ u(x, D, \bar{\sigma}) :\ (x, D, \theta, \varphi) \in B(p, \pi, K)\} \tag{1}$$

for his behavior. Here $u(x, D, \sigma)$ represents a utility function where $x = (x_0, \cdots, x_S)$ denotes one's consumption bundle; $D = (D_{sj}) \in \mathscr{R}^{S\times J}$, D_{sj} standing for the amount of default on asset j in state s;[6] θ (resp. φ) represents a purchase (resp. sales) portfolio of the J assets. $\bar{\sigma} = (p, \pi, K, \Lambda)$ where p is a vector of commodity prices, and π of asset prices; $K = (K_{sj}) \in \mathscr{R}^{S\times J}$, K_{sj} being the delivery rate of asset j in state s; $\Lambda = (\Lambda_j) \in \mathscr{R}^J$, $\Lambda_j = ja\lambda$ being the penalty rate of asset j, uniformly for all states; a, λ are parameters and λ will be termed *penalty level.* We assume throughout that the utility function takes the form

$$u(x, D, \sigma) = u_0(x) - \sum_{s\in\mathbb{S}} \sum_{j\in\mathbb{J}} q_s \Lambda_j D_{sj},$$

where $u_0(x)$ is a concave function of x, q_s the probability that state s occurs. Let $\mathscr{D}_s = \partial u/\partial x_s$.

ASSUMPTION 1. *(i) All households possess positive endowments in each state, or $\omega^i = (\omega^i_0, \cdots, {\omega^i}_S)^T \gg 0, i \in \mathbb{H}$, where a vector $\bar{\xi}$ is positive, denoted $\bar{\xi} \gg 0$, means each of its components is positive; (ii) Marginal utility $\mathscr{D}_s$ of commodity s is continuous in x, and $\lim\limits_{x_s\to 0} \mathscr{D}_s = +\infty, s \in \{0\} \cup \mathbb{S}$.*

[5]What we shall enter into discussion below applies also to an economy with more than one commodity yet the payoffs of assets still given in terms of one commodity.

[6]In Dubey et al. (2005), the matrix D indicates the amount of delivery one makes on asset j in state s.

The budget set, $B(p,\pi,K)$, is given by

$$\begin{aligned} B(p,\pi,&K) \\ &= \Big\{(x,D,\theta,\varphi) \in \mathscr{R}_+^{S+1} \times \mathscr{R}_+^{J\times S} \times \mathscr{R}_+^{J} \times \mathscr{R}_+^{J} : \\ &\qquad (x_0-\omega_0) + \pi^T(\theta-\varphi) = 0, && (2)\\ &\qquad (\bar{x}_0-\bar{\omega}_0) + R\varphi - D\mathbf{1} = (K\odot R)\theta, && (3)\\ &\qquad D - R\cdot \mathrm{dg}(\varphi) \le 0, && (4)\\ &\quad\Big\} \end{aligned}$$

where $\bar{x}_0 = (x_1,\cdots,x_S)^T$, $\bar{\omega}_0 = (\omega_1,\cdots,\omega_S)^T$; $K\odot R$ is the Hadamard product matrix of K and R;[7] $\mathrm{dg}(\bar{\xi})$ is an operator producing a diagonal matrix whose diagonal is formed by the components of vector $\bar{\xi}$. Both conditions (2) and (3) are homogenous of degree one in price; hence we can take $p_s = 1, s\in\{0\}\cup\mathbb{S}$. Condition (2) says that the expenditure on consumption and purchase of assets should equalize the revenue from selling endowment and assets; condition (3) means what one actually consumes and delivers to the market in state s should be equal to the sum of his endowment and what he get delivered from the market. Condition (4) requires one not to default on any asset in any state beyond what he could, and $D\in\mathscr{R}_+^{J\times S}$ implies none would make over-delivery (deliver more than he promises).

Let

$$x-\omega = \begin{bmatrix} x_0-\omega_0 \\ \bar{x}_0-\bar{\omega}_0\end{bmatrix};\quad y = \begin{bmatrix} x-\omega \\ d \\ \theta \\ \varphi\end{bmatrix} = \begin{bmatrix} x-\omega \\ \bar{y}\end{bmatrix};$$

and

$$M = \left[\begin{array}{cc:ccc} 1 & 0 & 0 & \pi^T & -\pi^T \\ 0 & I_S & -\tilde{I}_S & -K\odot R & R\end{array}\right] = \begin{bmatrix} I_{S+1} & M_1\end{bmatrix};$$

where $d = \mathrm{vec}(D)$, the operator *vec* transforming a matrix into a vector by stacking the columns of the matrix one underneath the other; I_S and I_{S+1} are, respectively, identity matrices of order S and $S+1$; $\tilde{I}_S = [I_S,\cdots,I_S]\in\mathscr{R}^{S\times(S\cdot J)}$.

[7]The Hadamard product $K\odot R$ is a matrix of the same dimension with K and R, and its element at spot (i,j) is given by $K_{ij}\cdot R_{ij}$.

To handle the boundary conditions and inequality constraints, we assume that a household, in quest of his behavior, optimizes

$$\begin{cases} \max\limits_{(x,d,\theta,\varphi)} & u(x,d,\sigma)+\phi(x,d,\theta,\varphi) \\ \text{s.t.} & My=0, \end{cases} \tag{5}$$

where $\phi(x,d,\theta,\varphi)=\beta\big[\sum_s \log x_s+\sum_s\sum_j(\log(R_{sj}\varphi_j-D_{sj})+\log D_{sj})+\sum_j\log\theta_j+\sum_j\log\varphi_j\big]$, with β a positive scalar;[8] and $\sigma=\mathcal{G}(\pi,K,\lambda)=\big[\pi^T\,(\mathrm{vec}K)^T\,\lambda\big]^T$. Here we define $\log(\xi)$, for any scalar ξ, as

$$\log(\xi)=\eta_1(\lambda)\ln\xi+\eta_2(\lambda)(-\frac{1}{2\varrho^2}\xi^2+\frac{2}{\varrho}\xi+\ln\varrho-\frac{3}{2}),$$

where ϱ is any small positive number, and take, given any two penalty levels λ_0 and λ_1 with $\lambda_0>\lambda_1$,

$$\eta_1(\lambda)=\begin{cases} a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4 & \lambda\in[\lambda_1,\lambda_0], \\ 1 & \lambda\in[0,\lambda_1], \end{cases} \qquad \eta_2(\lambda)=1-\eta_1(\lambda),$$

such that $\eta_1(\lambda_0)=0,\eta_1(\lambda_1)=1,\eta_1'(\lambda_1)=0,\eta_1''(\lambda_1)=0$, intended to obtain a function of second-order differentiability.[9] This definition brings forth much convenience in looking for the starting point of our algorithm. We claim that

Theorem 1. *The demand function (x,d,θ,φ) defined by (5) is differentiable with respect to σ; so is the indirect utility function $V(\sigma)$.*

Proof. Let $\bar{u}(x,d,\theta,\varphi,\sigma)=u(x,d,\sigma)+\phi(x,d,\theta,\varphi)$. By the first-order optimality condition, we have

$$\begin{cases} \nabla_y\bar{u}-\mu M^T=0, \\ My=0, \end{cases} \tag{6}$$

[8]Here $\phi(x,d,\theta,\varphi)$ can be thought of as a generalized Cobb-Douglas utility function. This function ensures that consumption x, default d, purchase portfolio θ, and sales portfolio φ are all strictly positive at equilibrium.

[9]Let $\eta_0(\lambda)=a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4$. We must see to it that $\eta_0(\lambda)$ belongs to $(0,1)$ when λ is in (λ_1,λ_0). In fact, $\eta_0'(\lambda)=(\lambda-\lambda_1)(3a_1\lambda+b_1)$, where b_1 is some constant. By

$$\eta_0''(\lambda)\Big|_{\lambda=\lambda_1}=(6a_1\lambda+b_1-3a_1\lambda_1)\Big|_{\lambda=\lambda_1}=3a_1\lambda_1+b_1=0,$$

we conclude that either $\eta_0'(\lambda)>0$ or $\eta_0'(\lambda)<0$ when λ is not equal to λ_1. But from $\eta_0(\lambda_0)=0,\eta_0(\lambda_1)=1$, we can only have $\eta_0'(\lambda)<0$, so $\eta_0(\lambda)$ is monotonically decreasing in (λ_1,λ_0).

where μ is a vector of Lagrangian multipliers. By assumption, $u(x, d, \sigma)$ is concave in x and linear in d, so its Hessian matrix $\nabla_y^2 u$ is negative semidefinite. By definition, $\phi(x, d, \theta, \varphi)$ is strictly concave in y, so $\nabla_y^2 \phi$ is also negative semidefinite. It is easy to check that $\nabla_y^2 \phi$ is strictly diagonally dominant, so it is nonsingular. Hence $\nabla_y^2 \phi$, and therefore $\nabla_y^2 \bar{u} = \nabla_y^2 u + \nabla_y^2 \phi$, is negative definite. Noting that M is of full rank, the Jacobian matrix of (6) with respect to y and μ

$$\begin{bmatrix} \nabla_y^2 \bar{u} & -M^T \\ \\ M & 0 \end{bmatrix}$$

is nonsingular. The theorem thus holds true, by the implicit function theorem. ▌

To acquire the demand function (x, d, θ, φ), it is convenient to eliminate from (5) the constraint $My = 0$. Solving it for x gives $x = \omega - M_1 \bar{y}$. Define $U(\bar{y}, \sigma) = \bar{u}(\omega - M_1 \bar{y}, d, \theta, \varphi, \sigma)$. Problem (5) is then tantamount to finding a

$$\bar{y}^*(\sigma) \in \arg\max\ U(\bar{y}, \sigma), \tag{7}$$

and the corresponding first-order optimality condition boils down to

$$\nabla_{\bar{y}} U(\bar{y}, \sigma) = 0. \tag{8}$$

Note that some assets may not be traded in problem (1); but all assets will each be traded in problem (5) when $\eta_1(\lambda)$ is positive. The coefficient β may thus assume the interpretation that the government intervenes to incite (or impel) the households to trade in all assets, and as it approaches zero this intervention will fade away, which is essentially in the spirit of *refined equilibrium* in Dubey et al. (2005).

2.2. The equilibrium

An equilibrium is defined to be a list $\{\pi, K, \lambda, (x^i, D^i, \theta^i, \varphi^i)_{i \in \mathbb{H}}\}$ such that

- $(x^i, D^i, \theta^i, \varphi^i)$ is the Marshallian demand function of household i, given (π, K, λ), where K is rational expectations of delivery rates;
- All markets clear. That is,

$$\sum_{i \in \mathbb{H}} (\theta^i - \varphi^i) = 0, \tag{9}$$

$$[(\mathbf{1} - K) \odot R] \cdot \mathrm{dg}(\sum_{i \in \mathbb{H}} \varphi^i) - \sum_{i \in \mathbb{H}} D^i = 0; \tag{10}$$

or, compactly,

$$Z(\sigma) = 0; \tag{11}$$

The first condition is saying that the market for assets is cleared; the second dictates that what households get delivered from the market be all equal to what they deliver to the market. By Walras's law, the market for commodities will automatically be cleared.

But, with default, solving (11) for an equilibrium is not adequate. Besides, the requirement of rational expectation on delivery rates has to be fulfilled at its solution, which mathematically equals, for $\forall s \in \mathbb{S}, j \in \mathbb{J}$,

$$\text{if } \mathscr{F}_{sj} < \mathscr{D}_s, \ \text{ then } D_{sj} = \mathscr{N}_{sj}, \tag{12}$$

$$\text{if } \mathscr{F}_{sj} > \mathscr{D}_s, \ \text{ then } D_{sj} = 0, \tag{13}$$

where $\mathscr{F}_{sj} = \partial u/\partial D_{sj}$, and $\mathscr{N}_{sj} = R_{sj}\varphi_j$. Condition (12) (resp. (13)) stipulates that one should default completely (resp. deliver fully) on any asset in any state, provided, in that state, marginal disutility of default on that asset falls short of (resp. exceeds) marginal utility of commodity. As problem (1) is, however, being approximated by problem (5), neither condition can be satisfied strictly, so we relax them slightly, for $\forall s \in \mathbb{S}, j \in \mathbb{J}$,

$$\text{if } \bar{\mathscr{F}}_{sj} < \bar{\mathscr{D}}_s - \kappa_1, \ \text{ then } \mathscr{N}_{sj} - D_{sj} \le \kappa_2, \tag{12$'$}$$

$$\text{if } \bar{\mathscr{F}}_{sj} > \bar{\mathscr{D}}_s + \kappa_1, \ \text{ then } D_{sj} \le \kappa_2, \tag{13$'$}$$

where $\bar{\mathscr{D}}_s = \partial \bar{u}/\partial x_s, \bar{\mathscr{F}}_{sj} = \partial \bar{u}/\partial D_{sj}$, and κ_1, κ_2 are arbitrary positive scalars. A solution to (11) satisfying (12′) and (13′) will be referred to as a κ-equilibrium with $\kappa = (\kappa_1, \kappa_2)$.

3. COMPUTE THE OPTIMAL PENALTY LEVEL

3.1. Optimal penalty level of a household

Household 1 ascertains his optimal penalty level by solving

$$\begin{cases} \max\limits_{\sigma} & V_1(\sigma) \\ \text{s.t.} & Z(\sigma) = 0. \end{cases} \tag{14}$$

The necessary condition for σ to be optimal takes on the form

$$0 = \delta_0 \nabla_\sigma V_1 + \sum_{k=1}^{\bar{k}} \delta_k \nabla_\sigma Z_k(\sigma), \tag{15}$$

where $\delta_k, k = 0, \cdots, \bar{k}$ are not all simultaneously zero and $\bar{k} = (S+1) \times J$. Geometrically, (15) asserts that, when δ_0 is nonzero, $\nabla_\sigma V_1$ belongs to the column space Ω of matrix $\mathbf{T} = [\nabla_\sigma Z_1(\sigma), \ \cdots, \ \nabla_\sigma Z_{\bar{k}}(\sigma)]$; in other words, the distance, denoted $\rho(\sigma)$, vanishes from $\nabla_\sigma V_1$ to Ω. Here, distance is defined, in terms of the Euclidean norm, as

$$\rho(\sigma) = \inf_{\varpi \in \Omega} \|\nabla_\sigma V_1 - \varpi\|,$$

where $\nabla_\sigma V_1$ is normalized to a unit vector by dividing its norm. Hence $\rho(\sigma)$ ranges between 0 and 1. Our goal in this section is to acquire all Fritz John points (feasible points satisfying (15)) of (14) along a particular path.

At times it is far from easy to solve (7) analytically for one's behavior. In this case, we have to resort to

$$\begin{cases} \max \quad U^1(\bar{y}^1, \sigma) \\ \ \text{s.t.} \quad Z(\sigma) = 0, \\ \qquad \nabla_{\bar{y}^i} U^i = 0, \quad i \in \mathbb{H} \end{cases}$$

or, compactly,

$$\begin{cases} \max \quad U^1(\bar{y}^1, \sigma) \\ \ \text{s.t.} \quad g(\bar{Y}, \sigma) = 0 \end{cases}$$

where $g: \mathscr{R}^{L+1} \to \mathscr{R}^L, L = (S \times J + 2J)\hbar + J + S \times J; \bar{Y} = \text{vec}([\bar{y}^1, \cdots, \bar{y}^\hbar])$. To obtain Fritz John points of (14), let us construct a homotopy

$$H_1(\bar{Y}, \sigma) = g(\bar{Y}, \sigma) + \eta_1(\lambda)\rho(\sigma) \cdot \alpha, \tag{16}$$

and we shall henceforth be after zero points of H_1, through a path-following method.

For this method to operate, four questions are at issue: provide an easy system with a unique solution, guarantee that the primary route exists and that it is trapped, and finally present $\rho(\sigma)$ explicitly.

For the first question, we choose to start from system (16) with a sufficiently large penalty level λ_0. To make its solution easily accessible, find a state in which assets are all with positive payoffs. Without loss of generality, let it be state 1. If no such state is available, define another set of assets whose payoff matrix $P(\lambda)$ is the same as R except in state 1. As to state 1, let $P_{1j}(\lambda) = R_{1j} + \eta_2(\lambda), \forall j \in \mathbb{J}$. For the endowment ω_1^i of household i in state 1, we modify it into $(\omega_1^i)'(\lambda) = \eta_1(\lambda)\omega_1^i, \forall i \in \mathbb{H}$, such that $(\omega_1^i)'(\lambda_0) = 0$ and $(\omega_1^i)'(\lambda) = \omega_1^i$ when $\lambda \leq \lambda_1$. Here λ_1 is chosen to be inside λ_0 and beyond $\max\limits_{i \in \mathbb{H}, s \in \mathbb{S}} \{\bar{\mathscr{D}}_s^i(\omega^i)\}$, where $\bar{\mathscr{D}}_s^i(\omega^i)$ is household i's marginal utility (by $\bar{u}^i(y)$) of commodity s at his endowment point. This assures us that one optimal penalty level (at least) would occur in $(0, \lambda_1]$. With respect to the value of λ_0, we ask it to be large enough—enough to deter each household from any economic activity. For household i, it is easy to see that he will offer not to default on asset j as long as, at $x^i = w^i, d^i = \mathbf{0}, \theta^i = \varphi^i = \mathbf{0}$,

$$\frac{\partial \bar{u}^i}{\partial D_{1j}} > \sum_{s \in \mathbb{S} \setminus \{1\}} \frac{\partial \bar{u}^i}{\partial x_s} + \frac{\partial \bar{u}^i}{\partial \theta_j} + \frac{\partial \bar{u}^i}{\partial \varphi_j};$$

that is, the cost of default in state 1 outweighs the total benefit in the rest states from any volume of trade in this asset. For the whole economy, such λ_0 will thus enable none to default on any asset in any state, that

$$\min_{\substack{i \in \mathbb{H} \\ j \in \mathbb{J}}} \frac{\partial \bar{u}^i}{\partial D_{1j}} > \max_{\substack{i \in \mathbb{H} \\ j \in \mathbb{J}}} \left\{ \sum_{s \in \mathbb{S} \setminus \{1\}} \frac{\partial \bar{u}^i}{\partial x_s} + \frac{\partial \bar{u}^i}{\partial \theta_j} + \frac{\partial \bar{u}^i}{\partial \varphi_j} \right\}. \tag{17}$$

And then $D^i_{sj} = 0, (\omega^i_1)'(\lambda_0) = 0$, and $P_{1j}(\lambda_0) > 0$ together imply that no one will sell any of the J assets, which, in turn, implies none will buy any of them, so the economy corresponding to λ_0 would indeed involve no economic activity at equilibrium. It is from such a trivial equilibrium that we wish to start. To make for this, however, we must ensure that it is regular and unique.

If the trivial equilibrium is irregular, add first the term $\eta_2(\lambda)(\mathscr{V} \odot X)$ to the right hand side of (16), where $\mathscr{V} \in R^{L+1}$ and $X = [\bar{Y}^T, \sigma^T]^T$, and then find the $\mathscr{V}$ of smallest 1-norm such that an $L \times L$ submatrix of the Jacobian matrix of H_1 at this trivial equilibrium is strictly diagonally dominant. With regard to the uniqueness, it turns out here to be sufficient to make determinate the equilibrium prices and delivery rates of assets. Alternatively, we add first the term $\eta_2(\lambda)(c_1 \odot \pi + c_2)$ to the left hand side of (9), and find $c_1, c_2 \in \mathscr{R}^J$ such that $\pi_0 = \mathbf{1}$ is the unique equilibrium asset prices; then redefine the delivery rates as

$$K_{sj} = 1 - \frac{\eta_2(\lambda) + \sum_{i \in \mathbb{H}} D^i_{sj}}{\eta_2(\lambda) + \sum_{i \in \mathbb{H}} R_{sj} \varphi^i_j}, \quad \forall s \in \mathbb{S}, j \in \mathbb{J},$$

which entitles $K = \mathbf{0}$ only to be the equilibrium delivery rates. With all these, suppose the resulting new Homotopy is denoted

$$\bar{H}_1(X) = \bar{g}(X) + \eta_1(\lambda)\rho(\sigma) \cdot \alpha.$$

Let $\bar{Y}_0 = \mathbf{0}$, $\sigma_0 = \mathcal{G}(\pi_0, \mathbf{0}, \lambda_0)$, and $X_0 = [\bar{Y}_0^T, \sigma_0^T]^T$. The Homotopy that we shall hereafter adopt, is given by

$$H(X) = \bar{g}(X) - \eta_2(\lambda)\bar{g}(X_0) + \eta_1(\lambda)\rho(\sigma) \cdot \alpha. \tag{16$'$}$$

Obviously, when $\lambda = \lambda_0$, the solution X_0 to $H = 0$ is now unique and regular.

For the second question, suppose λ^* is an optimal penalty level of household 1, and $J(H, \lambda)$ is the Jacobian matrix of H at λ. Three cases may arise:

1. $\rho(\sigma)$ is positive. On Sard's theorem, zero is a regular value of H for almost all $\alpha \in \mathscr{R}^L$.

2. $\rho(\sigma)$ vanishes, and $J(H, \lambda^*)$ is of full rank. Zero is still a regular value of H, and the hyperplane $\lambda = \lambda^*$ can be met transversely.

3. $\rho(\sigma)$ vanishes, but $J(H, \lambda^*)$ is not of full rank. To cross the hyperplane $\lambda = \lambda^*$ transversely, the term $\cos[\gamma(\rho)]\alpha$ is chosen to be added to the right hand side of (16), where

$$\gamma(\rho) := \begin{cases} -\pi[(\rho/\rho_0)^2 - 2\rho/\rho_0]/2 & \rho \in [0, \rho_0], \\ \pi/2 & \rho \in (\rho_0, 1], \end{cases}$$

a function leading the term to be dispensed with as soon as $\rho(\sigma)$ increases to ρ_0.

These three cases jointly have the existence of the primary route guaranteed.

For the third question, it suffices to show that the primary route is contained in a compact set. When $\eta_1(\lambda) > 0$, we have $x \gg 0, d \gg 0, \theta \gg 0, \varphi \gg 0$. Given the endowments of the households, x must be bounded from above. By the proof in Dubey et al. (2005) that an equilibrium exists, any equilibrium with $\|\varphi\|$ tending to infinity can be adapted to get another equilibrium with $\|\varphi\|$ finite. Without loss of generality, assume $\|\varphi\| < Q$. There then follows from (4) and (9) that d and θ are bounded from above. By the same token, π must be bounded from above. If, on the other hand, some π_j goes to zero, then the demand of θ_j will go to infinity; hence, $\pi \gg 0$. Finally, inequality (4) and equation (10) combine to place all K_{sj} in $[0, 1]$. So the primary route is trapped.

As to the last question, it requires computing $\nabla_\sigma V_1$ and $\nabla_\sigma Z_i$, $i = 1, \cdots, \bar{k}$. Recall that

$$V(\sigma) = \max_{\bar{y}} U(\bar{y}, \sigma),$$

so, by the envelope theorem,

$$\nabla_\sigma V = \nabla_\sigma U(\bar{y}, \sigma)\Big|_{\bar{y} = \bar{y}^*(\sigma)}.$$

For $\nabla_\sigma Z_i$, it suffices to compute $\nabla_\sigma \bar{y}$, and it is, due to the implicit function theorem, given by

$$\nabla_\sigma \bar{y} = -\left[\nabla^2_{\bar{y}} U\right]^{-1} \nabla^2_{\bar{y}\sigma} U. \tag{18}$$

The expression of $\rho(\sigma)$ can thus be rendered explicitly as:

$$\rho^2(\sigma) = (\nabla_\sigma V)'(I_{\bar{k}+1} - \mathbf{T}(\mathbf{T}'\mathbf{T})^{-1}\mathbf{T}')\nabla_\sigma V,$$

where $I_{\bar{k}+1}$ is the identity matrix of order $\bar{k}+1$.

Now a path presents itself; following it, solutions to (11) would result. But these solutions, as noted above, need not be equilibria, unless conditions (12′) and (13′) are fulfilled. Take Fig.1 for example. Suppose both Path I and Path II begin with point A (although only one of them is possible). Along Path I, as it is available, the desired equilibrium E^* would eventually be reached. Instead, if Path II emerges, what shall we do? Suppose at E_1 either condition (12′) or (13′) is violated. In this case, we adapt E_1 to obtain E_1', so as to advance, not necessarily along the line E_1E_2, toward E_2, and then build up a wall, to ensure that the Newton corrector, starting from E_1', will convey us to E_2, rather than going back to E_1. Conditions (12′) and (13′) are easy to check and pivotal is how to build up the wall. The ways to handle both conditions are alike and we shall next focus but on condition (12′).

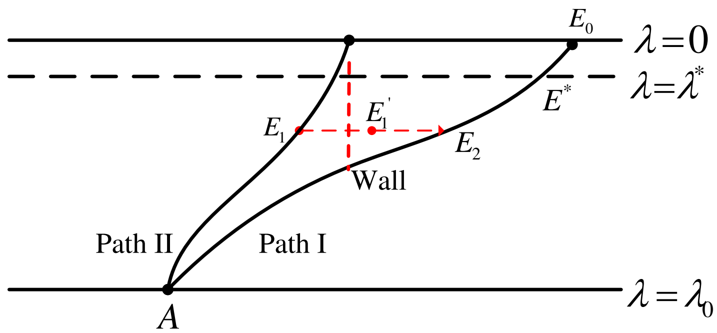

FIG. 1. Path I leads us to an equilibrium whereas Path II does not; and E^* is the desired equilibrium.

Observe that any violation of condition (12′) by household i implies:

$$\exists s \in \mathbb{S}, j \in \mathbb{J}, \text{such that } (i)\ \bar{\mathscr{F}}^i_{sj} < \bar{\mathscr{D}}^i_s - \kappa_1;\ (ii)\ \mathscr{N}^i_{sj} - D^i_{sj} > \kappa_2. \qquad (19)$$

For it anew to be satisfied, the default level D^i_{sj} should be raised to the extent, e.g., that $\bar{\mathscr{F}}^i_{sj} = \bar{\mathscr{D}}^i_s - \kappa_1$, or $\mathscr{N}^i_{sj} - D^i_{sj} = \kappa_2$. This would reduce the delivery rate of asset j in state s to

$$K'_{sj} = K_{sj} - \frac{\sum_{i \in \mathbb{H}_0} \zeta^i_{sj}}{\sum_{i \in \mathbb{H}} \mathscr{N}^i_{sj}},$$

where $\mathscr{N}_{sj}^i = \sum_{i\in\mathbb{H}} R_{sj}\varphi_j^i$, $\mathbb{H}_0$ is the set of households who violate condition $(12')$ and ζ_{sj}^i the increment in D_{sj}^i. Take φ_j^i as a function of ϑ, where $\vartheta = \text{vec}K$. At $\vartheta' = \text{vec}K'$, we have, to the first order approximation,

$$\varphi_j^i(\vartheta') \approx \varphi_j^i(\vartheta) + \sum_{k=1}^{S\times J} \frac{\partial \varphi_j^i(\vartheta)}{\partial \vartheta_k}(\vartheta_k' - \vartheta_k),$$

all partial derivatives evaluated by equation (18). The default levels at E_1' are given by, letting $\mathscr{N}_{sj}^i(\vartheta') = R_{sj}\varphi_j^i(\vartheta')$,

$$(D_{sj}^i)' = \mathscr{N}_{sj}^i(\vartheta') - \kappa_2, \text{ for } i \in \mathbb{H}_0; \;\; (D_{sj}^i)' = D_{sj}^i, \text{ for } i \notin \mathbb{H}_0;$$

and purchase portfolios given by

$$\theta_j^i(\vartheta') \approx \theta_j^i(\vartheta) + \sum_{k=1}^{S\times J} \frac{\partial \theta_j^i(\vartheta)}{\partial \vartheta_k}(\vartheta_k' - \vartheta_k), \text{ for } i \in \mathbb{H}.$$

Let $\bar{\mathscr{N}}_{sj} = \sum_{i\in\mathbb{H}} \mathscr{N}_{sj}^i$ and $\bar{\mathscr{N}}_{sj}' = \sum_{i\in\mathbb{H}} \mathscr{N}_{sj}^i(\vartheta')$. Define $\Delta_{sj} = [0, K_{sj}']$, if $\bar{\mathscr{N}}_{sj}' \leq \bar{\mathscr{N}}_{sj}$; and $\Delta_{sj} = [K_{sj}', K_{sj}'']$ otherwise; where $K_{sj}'' = K_{sj} - \epsilon$, ϵ being a small chosen number. Denote $\Delta = \prod_{s\in\mathbb{S}, j\in\mathbb{J}} \Delta_{sj}$. The following algorithm will be operating to yield an equilibrium from a given solution E_1 to (16).

Algorithm 1.

Step 1: Check whether conditions $(12')$ and $(13')$ are satisfied at E_1. If yes, stop, and output E_1.

Step 2: Adapt E_1 to get E_1'.

Step 3: Starting from E_1', use Newton method to explore Δ for a zero point $\bar{E}_1$ of Homotopy (16).

Step 4: Let $E_1 = \bar{E}_1$. Go to step 1.

This algorithm will generate a sequence $\{\bar{E}_k\}_{k\geq 1}$, and it can be proved that

Theorem 2. *This sequence converges to a κ-equilibrium.*

Proof. See the Appendix. ▌

3.2. Optimal penalty level of the economy

Let $\mathcal{E}(\lambda)$ be an equilibrium of the economy with penalty level λ, and $\mathbf{E} = \{\mathcal{E}(\lambda) : \lambda > 0\}$. By an optimal penalty level of the economy, is meant one leading to an equilibrium that is Pareto optimal in $\mathbf{E}$.

To obtain such penalty levels, we modify the homotopy (16′) into

$$\bar{H}(X) = \bar{g}(X) - \eta_2(\lambda)\bar{g}(X_0) + \eta_1(\lambda) \prod_{i\in\mathbb{H}} \rho_i(\sigma) \cdot \alpha, \tag{20}$$

and distinguish three cases, at $\lambda = \lambda^*$,

1. $J(\bar{H}, \lambda^*)$ is not of full rank,
2. $J(\bar{H}, \lambda^*)$ is of full rank and all $\rho_i(\sigma)$ are not zero,
3. $J(\bar{H}, \lambda^*)$ is of full rank and none of $\rho_i(\sigma)$ is zero.

Any penalty level λ^* conforming to the first two cases constitutes a candidate for being an optimal one of the economy. Identifying $\prod_{i\in\mathbb{H}} \rho_i(\sigma)$ in (20) with $\rho(\sigma)$ in (16), the method for calculating an optimal penalty level of the economy resembles that of a household.

3.3. Algorithm for computing the optimal penalty level of an economy

Aiming to find all Fritz John points along the primary route, we set the stopping criterion to be $\lambda \le \lambda_2$, where $\lambda_2 = \min_{i\in\mathbb{H}, s\in\mathbb{S}} \{\bar{\mathscr{D}}_s^i(\omega^i)\}$, a penalty level creating every household an incentive to default completely (on any asset in any state), and qualifying none of those penalty levels less than it to be optimal. As follows we sketch out the algorithm for computing the optimal penalty level of an economy. Let $\bar{\xi}(l)$ be the l-th component of vector $\bar{\xi}$, $e_k = \|\bar{H}(X_k)\|$ the error at point X_k, and $\bar{u}_k^i$ the utility level of household i corresponding to X_k.

Algorithm 2.

Step 0: Get the starting point X_0. Set $k = 0$, $\lambda = \lambda_0$, and $\mathbb{U}^i = \emptyset, i \in \mathbb{H}$.

Step 1: If $\lambda < \lambda_2$, then stop. Find the optimal penalty level from $\{\mathbb{U}^i, i \in \mathbb{H}\}$.

Step 2: Compute e_k and the tangent vector $\vec{v}_k$.

Step 3: Let $\mathbb{U}^i = \mathbb{U}^i \cup \{\bar{u}_k^i\}, i \in \mathbb{H}$.

Step 4: Call algorithm 0 to get X_{k+1}.

Step 5: Let $E_1 = X_{k+1}$. Call algorithm 0, and suppose it outputs X'_{k+1}.[10]

[10] In step 3 of algorithm 1, (16) should be changed into (20).

Step 6: Scale down $\{\beta^i : i \in \mathbb{H}\}$ by a factor less than one.
Step 7: Let $X_{k+1} = X'_{k+1}, \lambda = X_{k+1}(L+1), k = k+1$. Go to Step 1.

To the performance of a path-following algorithm, step length, as is known, contributes substantially. The dilemma here is that: we can neither stride lest the optimal penalty level be passed over, nor can we snail in order to shun a prohibitively large amount of computation. To strike a balance, the following way will be adopted.

Suppose X_0 is a zero point of Homotopy (20), and the tangent vector at X_0 is $\vec{\upsilon} = [\upsilon_1^T \ \upsilon_2]^T$, where $\vec{\upsilon}$ is normalized to be a unit vector and υ_2 is the last component of $\vec{\upsilon}$. If the absolute value of υ_2 falls below some threshold ς_1, let $\upsilon_2' = \mathrm{sgn}(\upsilon_2)\varsigma_2$, and $\vec{\upsilon}' = [\upsilon_1^T \ \upsilon_2']^T$, where *sgn* represents the sign function and ς_1, ς_2 are less than one. It can be proved that

THEOREM 3. *There exist a step length τ_0 and another zero point X^* of Homotopy (20) such that the Newton method will definitely converge to X^*, whether the starting point is $X_1 = X + \tau_0\vec{\upsilon}$ or $X_1' = X + \tau_0\vec{\upsilon}'$.*

Proof. See the Appendix. ▌

This theorem suggests the following manner of implementing the two major steps, prediction and correction, of a path-following algorithm.

ALGORITHM 3.

Step 0: Given X_k, $\vec{\upsilon}_k$, the step length τ, the error e_k, and $\varsigma_0, \varsigma_1, \varsigma_2$.
Step 1: Let $t = \vec{\upsilon}_k(L+1)$. If $|t| < \varsigma_1$, let $\vec{\upsilon}_k(L+1) = \mathrm{sgn}(t)\varsigma_2$.
Step 2: $X_k' = X_k + \tau\vec{\upsilon}_k$.
Step 3: Starting from X_k', use the Newton method to get zero point X_{k+1} of (20).
Step 4: Compute the error e_{k+1} at X_{k+1}.
Step 5: If $e_{k+1} > \varsigma_0 e_k$, let $\tau = \tau/2$, and go step 2.
Step 6: Output X_{k+1}.

3.4. Numerical experiment

The example used here is adapted from example 2 of Dubey et al. (2005). Their difference resides solely in that household i is endowed with 0.1, rather than 0, unit of commodity in state i, so as to obey assumption 1. For this economy, the optimal penalty level is 8/7; $x^1 = (0.35, 0.875, 0.875)^T$, $x^2 = (0.875, 0.35, 0.875)^T$, $x^3 = (0.875, 0.875, 0.35)^T$; $\theta^i = \varphi^i = 0.375$, and household i defaults completely in state i and delivers fully in the other two states, for all $i \in \mathbb{H}$; delivery rates are given by $K_s = 2/3$, for all $s \in \mathbb{S}$.

Take parameters of algorithm 0 as follows and the optimal penal-

β	ϱ	λ_0	λ_1	λ_2	ς_0	ς_1	ς_2	ρ_0	κ_1	κ_2	τ
10^{-3}	10^{-3}	25	15	1	2	0.1	0.1	0.1	0.1	0.1	0.05

ty level resulting from our algorithm is 1.2; the consumptions are given by $x^1 = (0.3315, 0.8876, 0.8855)^T$, $x^2 = (0.8875, 0.3315, 0.8854)^T$, $x^3 = (0.8816, 0.8816, 0.3296)^T$; the purchase and sale portfolios by $(\theta^1, \varphi^1) = (0.3526, 0.3501)$, $(\theta^2, \varphi^2) = (0.3526, 0.3503)$, $(\theta^3, \varphi^3) = (0.3532, 0.3572)$, and delivery rates by$(K_1, K_2, K_3) = (0.6575, 0.6573, 0.6509)$.[11] This solution is close to the true optimal solution, demonstrating the effectiveness of the algorithm proposed.

4. CONCLUSION

Aside from consumption and production, default can also be regarded as a major economic activity. One of the long established ideas to inhibit such an activity is penalty; whose level, as is well-known, is closely allied with the market efficiency. In this paper, we develop a path-following algorithm to compute the penalty level that yields an equilibrium of Pareto efficiency. Numerical experiment shows that this algorithm is both effective and efficient. Another idea to inhibit the default behavior is collateral—asking the seller of an asset to put up collateral to back up his promise. Again the level of collateral bears profoundly on the market efficiency, and the analysis of this bearing constitutes our future work.

APPENDIX A

PROOF OF THEOREM 2

Observe that, whatever it is, Δ_{sj} is contained in $[0, K''_{sj}]$. So it is sufficient to prove the convergence of algorithm 0 when $[0, K''_{sj}]$ is explored for an equilibrium. Suppose the rational expectation of delivery rate on asset j in state s is $\tilde{K}_{sj}$.

Under assumption 1, the consumption x_s of a household in state s must be within a closed interval, say $\mathcal{I}$; so $\bar{\mathscr{D}}_s$, viewed as a function of x_s, is

[11] One needs to solve $\{\max_{\bar{y}} U(\bar{y}) : \ \pi(\theta - \varphi) = 0\}$ for $\bar{y}$, and cannot plug $\theta = \varphi$ into $U(\bar{y})$, because θ and φ yield different levels of marginal utility when delivery rates are not all equal to unity.

uniformly continuous on $\mathcal{I}$. That is, for κ_1, there exists some positive number κ_3 such that for all x_s, x'_s in $\mathcal{I}$ with $|x_s - x'_s| < \kappa_3$, the value of $\bar{\mathscr{D}}_s$ satisfies

$$|\bar{\mathscr{D}}_s(x_s) - \bar{\mathscr{D}}_s(x'_s)| < \kappa_1.$$

Suppose at x_s^0, $\bar{\mathscr{F}}_{sj} = \bar{\mathscr{D}}_s$. It follows from (i) of (19) that $x_s^0 - x_s > \kappa_3$, which, coupled with (ii) of (19), yields $\zeta_{sj} > \kappa_0 = \min\{\kappa_2, \kappa_3\}$. Take $\epsilon = \frac{\kappa_0}{\hbar Q R_{sj}}$. Then $\tilde{K}_{sj}$ must be within $[0, \mathscr{U}_1]$, where $\mathscr{U}_1 = K''_{sj}$. Hence, to $\{\bar{E}_k\}_{k\geq 1}$ generated by algorithm 0 there corresponds a monotonically decreasing sequence $\{\mathscr{U}_{k_i}\}_{i\geq 1}$ with $\mathscr{U}_{k_{i+1}} = \mathscr{U}_{k_i} - \epsilon$, $\mathscr{T}_1 = \{k_i\}_{i\geq 1}$ a subsequence of $\{k\}_{k\geq 1}$, and $\tilde{K}_{sj} \in [0, \mathscr{U}_{k_i}]$.

Likewise, a monotonically increasing sequence $\{\mathscr{L}_{\tilde{k}_r}\}_{r\geq 1}$ would result with $\mathscr{L}_{\tilde{k}_{r+1}} = \mathscr{L}_{\tilde{k}_r} + \epsilon$, $\mathscr{T}_2 = \{\tilde{k}_r\}_{r\geq 1}$ another subsequence of $\{k\}_{k\geq 1}$, disjoint with $\mathscr{T}_1$, and $\tilde{K}_{sj} \in [\mathscr{L}_{\tilde{k}_r}, K''_{sj}]$, if condition (13′) is violated in algorithm 0.

Altogether, there must exist an $l \in \mathscr{T}_1$ and $\tilde{l} \in \mathscr{T}_2$, such that, by exploring the interval $[\mathscr{L}_{\tilde{l}}, \mathscr{U}_l]$, a solution to (16) satisfying (12′) and (13′) (therefore a κ-equilibrium, by definition) will be obtained. □

PROOF OF THEOREM 3

The zero point X of Homotopy (20) is a continuous function of λ, denoted $X = X(\lambda)$. Suppose $X_0 = X(\bar{\lambda}_0)$ and the Newton method converges at $X(\lambda)$ for all $X' \in O(X(\lambda), \varepsilon(\lambda))$. From continuity, it follows that

$$\lim_{\lambda \to \bar{\lambda}_0^-} \|X_0 - X(\lambda)\| = 0;$$

so there must exists a λ^*, such that

$$\|X_0 - X(\lambda^*)\| < \frac{\varepsilon(\lambda^*)}{2} - \tau_0,$$

where τ_0 is taken to be less than $\varepsilon(\lambda^*)/2$. Due to $\|\vec{v}\| = 1$, we have

$$\|X_0 + \tau_0\vec{v} - X(\lambda^*)\| \leq \|X_0 - X(\lambda^*)\| + \tau_0 < \frac{\varepsilon(\lambda^*)}{2}.$$

Then
$$\begin{aligned} &\|X_0 + \tau_0\vec{v}' - X(\lambda^*)\| = \|X_0 + \tau_0\vec{v} - X(\lambda^*) + \tau_0\vec{v}' - \tau_0\vec{v}\| \\ \leq &\|X_0 + \tau_0\vec{v} - X(\lambda^*)\| + \tau_0\|\vec{v}' - \vec{v}\| \leq \frac{\varepsilon(\lambda^*)}{2} + \tau_0\varsigma_2 \\ < &\varepsilon(\lambda^*), \text{ by } \varsigma_2 < 1. \end{aligned}$$

Hence $X_0 + \tau_0 \vec{v}' \in O(X(\lambda^*), \varepsilon(\lambda^*))$. $\square$

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