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# Liquidity, Interest Rates, and Output

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This paper integrates monetary search theory with limited participation to analyze the liquidity effect of open market operations. The model features a centralized bonds market with limited participation and a decentralized goods market with random matches. In a fraction of matches, buyers can use unmatured bonds together with money to purchase goods. In other matches, a legal restriction forbids the use of bonds as the means of payments. In this economy, a shock to bond sales has two distinct liquidity effects. One is the immediate liquidity effect on the bond price and the nominal interest rate. The other is a liquidity effect in the goods market starting one period later, which arises as unmatured bonds facilitate trades. Thus, even independent shocks in the open market can have persistent effects on interest rates and real output. I establish the existence of the equilibrium and, with numerical examples, examine equilibrium properties.

*Key Words*: Liquidity; Interest rates; Money; Search.
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## 1. INTRODUCTION

This paper integrates monetary search theory with Lucas' (1990) model of limited participation to analyze the liquidity effect of open market op-

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erations. I show that shocks to government bond sales can affect interest rates and output persistently even when the shocks are independent over time. These shocks also affect the term structure of interest rates.

The liquidity effect of open market operations refers to the effect that an unexpected purchase of government bonds by a central bank reduces the nominal interest rate and an unexpected sale of government bonds drives up the interest rate. This effect is intuitively related to liquidity for the reason that when a central bank uses money to purchase government bonds, it injects liquid assets (money) into the market to replace relatively less liquid assets (bonds). The liquidity effect is separate from the effect of open market operations on expected inflation, because it can arise even when a central bank uses other policies in conjunction with open market operations to maintain a constant money growth rate. Moreover, the liquidity effect has real consequences. As Christiano et al. (1999) documented with vector autoregression, even a temporary monetary shock through open market operations affects real activities persistently. Thus, analyzing the liquidity effect is important for understanding the effect of monetary policy.

For open market operations to generate the liquidity effect, an economy must have two necessary features. First, money must have higher liquidity than bonds in the sense that money is more readily accepted in the exchange for goods. If the two assets were equally liquid, swapping one for the other would have no effect on the liquidity of the asset portfolio. Second, there must be frictions that delay the transmission of the newly injected money from the bond market to the goods market. If the injection could immediately spread to the goods market, instead, the price level of goods would increase immediately, which would lead to an increase rather than a decrease in the nominal interest rate. In an influential paper, Lucas (1990) constructs a model with these two features. In his model, the lower liquidity of bonds relative to money comes from the assumption that a buyer can only use money to buy goods. This cash-in-advance constraint can be interpreted as a legal restriction. To prevent money from immediately flowing from the bond market to the goods market, Lucas (1990) assumes that the two markets are separated from each other in a period. This separation can be interpreted as a result of a participation cost in the bond market. Because of this feature, Lucas' model is often referred to as a model of limited participation. In this model, Lucas demonstrates that a pure liquidity effect arises. Specifically, when the government unexpectedly sells bonds for money and uses a lump-sum monetary tax/transfer to keep the money growth rate constant, the nominal interest rate rises.

The liquidity effect in Lucas' model is not persistent. That is, if the shocks in open market operations are independent over time, then the nominal interest rate and real activities depend only on the contemporaneous shock but not on past shocks. This prediction of the model contrasts

with the evidence of vector autoregression mentioned earlier. Since Lucas' work, a large literature has attempted to generate a persistent liquidity effect, with limited success (see Christiano et al., 1999, for a partial list of references). Most of the models in this literature add impediments to the flow of funds from the bonds market to the goods market but maintain the assumption that the goods market is Walrasian with a cash-in-advance constraint. In this paper, I impose the same assumption on limited participation in the bond market as Lucas does, but make two changes to the modelling of the goods market. First, the goods market is not Walrasian; instead, it is decentralized and characterized by search and matching. Second, the legal restriction that prevents bonds from being used as a medium of exchange is partial; that is, it is enforced in only a fraction of the trades rather than all trades.

The main motivation for modelling the goods market as search and matching is to provide a microfoundation of money. Originated in Kiyotaki and Wright (1989), monetary search theory generates a role of money endogenously from the trading frictions in the goods market, even when money has no intrinsic value in preferences and technology.[1] There is an additional motivation for such modelling. When the exchange in the goods market is decentralized, it is natural to assume that funds in the bond market take time to flow to the goods market, as is required for the liquidity effect to arise. Although monetary search theory is appealing for these reasons, it has largely been formulated in deterministic settings, rather than the stochastic setting that macroeconomists calibrate to investigate the statistical relationships between aggregate variables. For the liquidity effect of open market operations, in particular, a stochastic environment is necessary because the effect often does not arise in deterministic environments (see Lucas, 1990).

The motivation for modelling the legal restriction as a partial rather than complete restriction is to allow shocks in open market operations to affect the composition of assets used in the exchange in the goods market. As explained below, this change in the composition induces the liquidity effect to be persistent. Although government bonds are rarely used directly in the exchange for goods, it is not difficult to find the evidence on the indirect use of bonds in the goods market. For example, government bonds with short maturity are a large part of money market checking accounts on which checks are written and other means of payment are created to pay for goods. Although checking accounts and other means of payment are products and services provided by financial institutions, I abstract from

[1]Examples of search models are Trejos and Wright (1995), Shi (1995, 1997), Green and Zhou (1998).

the detailed modelling of financial intermediation and, instead, assume that bonds are partially liquid in the goods market.

Let me describe the model briefly. The economy has a bond market and a goods market, which are separated in each period. The bond market is centralized and functions in the same way as in Lucas (1990), where the government sellsnominal bonds at the market price and accepts only money as payments.[2] The amount of new bonds is stochastic, which is the only aggregate uncertainty in the economy. This shock is realized after the households have already allocated the assets between the two markets, and hence there is limited participation in the bond market. In contrast to the bond market, the goods market is decentralized; that is, matching is random and bilateral, barter is difficult, and individuals are anonymous. There are two types of matches. One is unrestricted matches, where the buyer can use both money and bonds to buy goods. The other is restricted matches, where a legal restriction forbids the use of bonds as the means of payments for goods. Restricted matches are a fraction $g \in (0, 1)$ of all matches. With this partial legal restriction, bonds are redeemed immediately at maturity, but unmatured bonds can circulate as an imperfect substitute for money. I set the bonds' maturity to be two periods — the shortest length that allows unmatured bonds to circulate in the goods market.

Open market operations in this model generate a delayed liquidity effect in the goods market, in addition to the immediate liquidity effect in the bond market. In particular, a high shock to bond sales in the previous period increases the quantity of unmatured bonds circulating in the current goods market. These additional bonds provide liquidity to the buyers who are in unrestricted trades. Thus, shocks to bond sales in the previous period change the current dispersion of real quantities of goods produced and traded in unrestricted matches versus restricted matches. As a result, aggregate output depends on the shock in the previous period even though shocks are independent over time. This persistence of the liquidity effect is absent in Lucas (1990). In addition, the model can generate reasonable serial correlations in output and a non-trivial term structure, as summarized in the concluding section.

The persistence of the liquidity effect depends on the assumption that the legal restriction is only imposed in a fraction of the trades in the goods market. If every trade were restricted (i.e., if $g = 1$), the model's predictions would resemble Lucas' (1990). On the other hand, if every trade were unrestricted (i.e., if $g = 0$), shocks in open market operations would change prices of goods uniformly across matches. In either case, there would be

[2] I maintain this cash-in-advance constraint in the bonds market in order to compare the results with Lucas'. However, it should be clear that this constraint is not the reason why money is valued here.

no delayed liquidity effect in the goods market and so independent shocks would not have persistent effects. In particular, the case $g = 0$ is not interesting for the current purpose.

A natural question is why it is necessary to construct a model with decentralized exchange in the goods market. Would a modification of Lucas' model with a partial legal restriction generate the same results as the current model? The answer is no. When the goods market is Walrasian, as in Lucas' model, agents can arbitrage between trades. As a result, matured bonds can circulate as a medium of exchange despite the partial legal restriction (see Shi, 2005). The circulation of matured bonds will reduce or even eliminate the real effects of open market operations. To obtain non-negligible real effects in such a model, one must assume either that $g$ is close to one or that matured bonds cannot circulate. The former is unrealistic and the latter is arbitrary. The current model does not need such assumptions. With decentralized exchange in the goods market, even an arbitrarily small coverage of the legal restriction is sufficient to induce agents to redeem all matured bonds immediately at maturity (see Shi, 2005).

One may also wonder whether the legal restriction can improve welfare (see Kocherlakota, 2003). In a related model, I have shown that a partial legal restriction can increase steady state welfare (Shi, 2008). I abstract from this welfare analysis in this paper.

As said above, there is a large literature on limited participation in which the goods market is assumed to be Walrasian. Deviating from this literature, Williamson (2006, 2008) has constructed models of limited participation with detailed modelling of the frictions in the goods market. The mechanism of monetary propagation in his models is different. In his models, there is a persistent separation or lack of "connection" between individuals that slows down the diffusion of the money injection among the population. As a result, even a perfectly anticipated money injection has long lasting real effects in his model, which seems implausible. In contrast, the propagation mechanism in my model works through the amount of "1iquidity" that unmatured bonds provide in the goods market. An anticipated money injection does not have any real effect in my model. Another noticeable difference of my model from Williamson's is the presence of aggregate uncertainty regarding monetary policy. Such uncertainty is necessary for characterizing the stochastic relationships between endogenous variables and monetary policy.

## 2. A MONETARY ECONOMY WITH SEARCH AND LEGAL RESTRICTIONS

In this section I describe an economy with a legal restriction in the goods market, analyze individuals' decisions, and define the equilibrium.

### 2.1. Households, Matches, and Markets

The economy has discrete time and many types of households. The number of households in each type is large and normalized to one. The households in each type are specialized in producing a specific good, which they do not consume, and exchange for consumption goods in the market. Goods are perishable between periods. The utility of consumption is $u(.)$ from consumption goods and 0 from other goods. The function $u$ is strictly increasing, concave and twice continuously differentiable, with the properties $u'(0) = \infty$ and $u'(\infty) < \infty$. The disutility of production is $\psi(.)$. To simplify the algebra, I will use the form $\psi(q) = \psi_0 q^{\Psi}$, where $\Psi > 1$ and $\psi_0 > 0$.

Each household consists of a large number of members, whose measure is normalized to one. The household makes all the decisions and the members simply carry out these decisions. The members regard the household's utility as the common objective. A fraction $\sigma$ of the members are sellers and the remaining are buyers, where $\sigma \in (0, 1)$. A seller produces and sells goods, while a buyer purchases goods. At the end of each period, the household pools all assets and goods received from trades, and then allocates the same amount of consumption to every member. As a result, individual matching risks are smoothed out within each household, and the distribution of asset holdings across households is degenerate. This degeneracy maintains tractability as it enables me to focus on the equilibrium that is symmetric across households.[3]

There are two assets in the economy — money and nominal bonds issued by the government. These assets can be stored without cost. Both have zero intrinsic value; i.e., they yield no direct utility or productive capacity. Nominal bonds are default-free and their maturity is two periods. A bond before the maturity is called an *unmatured bond*. Each bond can be redeemed for one unit of money at maturity. An alternative to redemption is to use matured bonds as a medium of exchange in the goods market in the future. Nothing in a traditional monetary model could prevent matured bonds from circulating as money. In contrast, when the exchange in the goods market is decentralized, households will always choose to redeem bonds immediately at maturity (see the introduction). In light of this result, it is innocuous to assume that matured bonds cannot be redeemed once they pass the maturity.

Let me describe the goods market first. In this market, agents meet trading partners bilaterally and randomly. Of interest are trade matches, in which the buyer likes the seller's goods. These matches are the only

[3]The large household is meant to approximate an agent's time allocation in different activities during a period. This modelling device, used by Shi (1997), is extended from a similar one in Lucas (1990).

ones in which a trade can take place. A buyer encounters a trade match at rate $\alpha\sigma$, and a seller at rate $\alpha(1-\sigma)$, where $\alpha < 1$. The total number of trade matches that all buyers (or sellers) of a household have in a period is $\alpha\sigma(1-\sigma)$. There is no double coincidence of wants to support barter, nor public record-keeping of transactions to support credit trade in the goods market. As a result, every trade entails a medium of exchange, which can be money or bonds. The strong assumptions on the matching patterns and record-keeping abilities are not necessary for the role of money. As is well established in monetary search theory (see the references cited in the introduction), fiat money continues to have a positive value even when there are bilateral credits, limited barter, multi-lateral meetings and imperfect public memory. The strong assumptions are imposed here to simplify the analysis and sharpen the focus on the competition between money and unmatured bonds as the media of exchange.

There are two types of matches. One is an *unrestricted trade*, where the buyer can use both money and bonds to buy goods. The other is a *restricted trade*, where a legal restriction requires money to be the only means of payments. The legal restriction is imposed in a fraction $g \in (0,1)$ of matches. One interpretation of the legal restriction is that a fraction $g$ of all agents are government agents who face the same matching rates as private agents but who accept only money as payments. Although I will use this interpretation later in the numerical exercises, I do not explicit model government agents here (for such modelling in a deterministic environment, see Shi, 2005). Notice that both restricted and unrestricted trades are decentralized exchanges.[4]

I model the legal restriction as a matching shock that occurs in each period after buyers and sellers have gone to the market. To be precise, suppose that there are a large number of locations in the goods market and each household goes to a particular location to search for matches. In each period, a government randomly selects a fraction $g$ of the locations to monitor and impose the legal restriction. If a location is monitored, all matches in the location face the legal restriction. This way of modelling the matching shock simplifies the analysis and has the following implications. First, in any given period, all of a household's buyers and sellers are involved in either restricted matches or unrestricted matches, but not in both. Second, because the shock occurs to matches, the two individuals in a match necessarily receive the same realization of the matching shock. This and the previous feature together imply that the two households in a match are symmetric, which simplifies the calculation of the terms of trade.

[4]One may suggest the alternative setup where restricted trades are Walrasian market and unrestricted trades are decentralized. As said above, this setup cannot prevent matured bonds from circulating as money, unless the government sector is sufficiently large.

The shock induces different quantities of trade and consumption between a household in restricted matches and a household in unrestricted matches. However, this ex post heterogeneity among the households is not carried over time. As I will explain in section 2.4, all households will optimally choose to hold the same portfolio of assets at the end of each period regardless of the matching shocks in the period. Thus, a representative household can be maintained over time.

In contrast to the goods market, the bond market has no transaction cost and trades take zero measure of agents. In this market, the government conducts open market operations by selling new bonds at the competitive price. As in Lucas (1990), the government only accepts money as payments for the bonds. However, agents can bring unmatured bonds into the bond market, sell them to other households for money, and then use the receipt to purchase new bonds, although the net amount of such transactions is zero in any symmetric equilibrium.

The amount of newly issued bonds is stochastic, which is the only aggregate uncertainty in the economy. To specify this stochastic process, let $M_{+1}$ be the average amount of money holdings per household in the *next period after* monetary transfers in that period are made but before the markets open (see Figure 1 later for a depiction of the timing). The amount of bonds newly issued in the current period is $zM_{+1}$, where $z$ is a random variable following a Markov process.[5] The realizations of $z$ lie in a finite set $Z$, with a lower bound $z_L > 0$ and an upper bound $z_H < \infty$. The transition function of $z$ is $\Phi(dz, z_{-1})$, where the subscript $-1$ indicates the previous period. Assume that $\Phi$ has the Feller property, i.e., that $f : Z \times Z \to R$ is continuous implies that $\int f(z, z_{-1})\Phi(dz, z_{-1})$ is continuous.

Open market operations can affect the money growth rate. However, to focus on the "pure" liquidity effect, Lucas (1990) eliminates the effect of the shocks on money growth in some sections of his paper by assuming that the government uses lump-sum transfers to maintain a constant money growth rate. In most parts of my analysis, I will adopt this assumption in order to compare my results with Lucas'. However, I will allow the money growth rate to vary with the shocks in section 6. Denote the gross rate of money growth as $\gamma$.

[5]The purpose of specifying the amount of new bonds as $zM_{+1}$, rather than $zM$ as in Lucas (1990), is for the convenience of unifying the formulas in the case where open market operations affect the money growth rate and the case where the money growth rate is fixed by monetary transfers (see a discussion in the next paragraph). The two specifications are equivalent, up to rescaling, in the case where the monetary growth rate is constant. Notice that the specification of $zM_{+1}$ does not create any problem of measurability. Because there is no new shock between the realization of the current $z$ and the measurement of future money stock $M_{+1}$, the stock $M_{+1}$ and the amount of newly issued bonds are measurable with respect tothe stochastic process of $z$.

### 2.2. Timing of Events

Let me clarify four pieces of notation. First, following Lucas (1990), I normalize all nominal quantities by the aggregate money holding per household, $M$. Second, I pick an arbitrary household as the representative household and use lower-case letters to denote the decisions of this household. The corresponding capital-case letters denote other households' decisions or aggregate variables. Third, I suppress the generic time subscript $t$, denoting $t \pm j$ as $\pm j$ for $j \geq 1$. Fourth, all integrals in this paper are over $Z$, with $Z$ being suppressed.

Figure 1 depicts the timing of events in each period. At the beginning of the period the household redeems bonds that were issued two periods ago and receives a lump-sum monetary transfer, $L$. After these events, the household's holding of money (divided by $M$) is measured as $m$, and of unmatured bonds as $b$.

| $t$ redeem | | portfolio $(a, \ell)$ | markets open; $z$ is realized | markets closed | $t+1$ |
|---|---|---|---|---|---|
| $\vert\!\longrightarrow$ | $\longrightarrow$ | $\longrightarrow$ | $\longrightarrow$ | $\longrightarrow$ | $\vert\!\rightarrow$ |
| money transfer $L$ | $(m, b)$ measured | decisions on goods trade $(q, x)$ | trades: $\gamma d, b^u$ and $(q, x)$ | pooling, consume | |

Figure 1 Timing of events in a period

Then, the household chooses a fraction of money, $a$, and a fraction of unmatured bonds, $\ell$, that will be taken to the goods market. This part of the assets the household divides evenly among the buyers; so, each buyer carries $am/(1-\sigma)$ units of money and $\ell b/(1-\sigma)$ units of unmatured bonds. The household takes the remaining assets to the bond market. At the time of choosing the portfolio divisions $(a, \ell)$, the household also chooses the quantities of goods and money for each buyer to offer in a trade (see the detailed description later). These quantities are contingent on whether the household members will be located in restricted or unrestricted trades. To indicate this contingency, I denote the quantities in a restricted trade as $(q^i, x^i)$, where $i = g$ indicates a restricted trade and $i = n$ indicates an unrestricted trade.

Next, the two markets open simultaneously and separately. It is not possible to communicate between the two markets. In the goods market, the matching shock is realized, which determines whether the household's members are in restricted or unrestricted trade. In each trade, the buyer makes a take-it-or-leave-it offer prescribed by the household as $(q^i, x^i)$. In the bond market, the shock $z$ is realized as the government issues an amount $\gamma z$ of new two-period bonds. Let $\gamma d$ be the amount of such bonds

demanded by the household, where $d$ is normalized by the aggregate money stock in the same way as $z$ is. The household can also trade unmatured bonds in the bond market. Let $b^u$ be the amount of unmatured bonds that the household carries out of the bond market when the market closes. Let the price of two-period bonds be $S$ and the price of unmatured bonds be $S^u$.

After the trades, the markets close and agents go home. The household pools the receipts from the trades and allocates consumption evenly among all members. After consumption, time proceeds to the next period.

As in Lucas (1990), the temporary separation between the two markets captures the costs of transferring assets and information between the two markets within the same period. These costs are critical for the liquidity effect, although the extreme form (i.e., infinite cost) may not be necessary. If agents could costlessly move assets between the two markets immediately after observing the shock in the bond market, or if the traders in the goods market can costlessly make the trades contingent on the realization of the current shock, then expected inflation would respond to the shock immediately. This would change the nominal interest rate in the way indicated by the Fisherian equation, which is opposite to the liquidity effect.

To make explicit these restrictions imposed by the temporary separation, I require that the portfolio divisions, $(a, \ell)$, and the quantities of trade in the goods market, $(q^i, x^i)$, be all independent of the current shock $z$, although they can depend on the past shock $z_{-1}$. In contrast, the amounts of bonds traded, $(d, b^u)$, can depend on $z$ as well as on $z_{-1}$, because the household chooses these amounts after observing the shock $z$. Similarly, prices of bonds, $(S, S^u)$, depend on both the shocks in the current period and in the previous period.

With the above timing, one-period bonds (if they are introduced) do not have a chance to circulate in the goods market before maturity. Once matured, they will be redeemed immediately by the households, rather than being kept to buy goods in a fraction of future trades. Thus, only unmatured long-term bonds can circulate in the goods market.

### 2.3. Quantities of Trade in the Goods Market

In a trade match, the buyer makes a take-it-or-leave-it offer. The household chooses the quantities of money and goods for each buyer to offer. To describe these choices, let $v(m, b, z_{-1})$ be the household's value function at the time where $m$ and $b$ are measured (see Figure 1).[6] The discount factor is $\beta \in (0, 1)$. Let $\omega^m(z_{-1})$ be the expected shadow value of next period's money discounted to the current period, where the expectation is calculated before observing the current shock $z$. Similarly, let $\omega^b(z_{-1})$ be

[6] I suppress the dependence of the value function on aggregate variables and other households' decisions.

the expected shadow value of unmatured bonds. Then,

$$\omega^i(z_{-1}) = \frac{\beta}{\gamma}\int v_{i_{+1}}(m_{+1}, b_{+1}, z)\Phi(dz, z_{-1}), \; i = m, b, \tag{1}$$

where $v_{i_{+1}} = \partial v(m_{+1}, b_{+1}, z)/\partial i_{+1}$. Notice that the discount on the value of future assets involves the money growth rate $\gamma$, because the variables $m$ and $b$ are normalized by the aggregate money stock which grows at rate $\gamma$. The expected values, $\omega^m$ and $\omega^b$, are computed before the current shock $z$ is realized, in order to make them relevant for the money allocation in the current period. Other households' expected value of future money is $\Omega^m$ and of future unmatured bonds is $\Omega^b$.

The offer specifies the quantity of goods that the buyer asks the seller to supply, $q$, and the quantity of assets that the buyer gives, $x$. These quantities are $(q^g, x^g)$ in a restricted trade and $(q^n, x^n)$ in an unrestricted trade. In a restricted trade, money is the only asset that can be used for the purchase. In an unrestricted trade, both money and unmatured bonds can be used. However, it is not necessary to specify the division of the amount $x^n$ into money and unmatured bonds, because the two assets are equivalent to anyone who exits the trade with them. When the trade is closed, no one can use these assets to purchase goods in the current period and, at the beginning of the next period, the bonds mature and can be redeemed for money at par.[7]

Under the assumption that the buyer makes a take-it-or-leave-it offer, the quantities $(q^i, x^i)$ yield zero surplus for the seller.[8] The seller's surplus in a type $i$ trade is $[\Omega^m x^i - \psi(q^i)]$, where $\Omega^m x^i$ is the value of the assets that the seller receives from the trade and where $i = n, g$. In this surplus, the seller's money receipts are evaluated at the margin because the contribution of one seller to the household's total money receipts is marginal. Notice that a household's sellers cannot share the production cost because production is incurred in each match. Setting the surplus to zero yields:

$$x^i(z_{-1}) = \frac{\psi(q^i(z_{-1}))}{\Omega^m}, \; i = n, g. \tag{2}$$

Also, the buyer is constrained by the sum of money and unmatured bonds in an unrestricted trade, and by the amount of money in a restricted trade. These *asset constraints* are:

$$x^n(z_{-1}) \leq \frac{a(z_{-1})m + \ell(z_{-1})b}{1-\sigma}, \tag{3}$$

[7]For the same reason, a trade in the goods market between a money holder and a bond holder is inconsequential, and so it is omitted here. Of course, this simplicity would be lost if bonds had maturity longer than two periods.

[8]See Shi (2001) for a formulation of sequetial bargaining that is embedded in a market with decentralized exchange and whose outcome coincides with that of Nash bargaining.

$$x^g(z_{-1}) \leq \frac{a(z_{-1})m}{1-\sigma}. \tag{4}$$

When an asset constraint binds, I say that the asset yields *liquidity services* in the goods market. Similarly, money may generate liquidity in the bond market.

### 2.4. A Household's Decision Problem

In a typical period, the household's choices are the portfolio division, $(a, \ell)$, the quantities of trade, $(q^n, x^n, q^g, x^g)$, the amount of new bonds to purchase, $\gamma d$, the amount of unmatured bonds exiting the bond market with, $b^u$, consumption, $(c^n, c^g)$, future money holdings, $m_{+1}$, and future holdings of unmatured bonds, $b_{+1}$. The decisions $(a, \ell, q, x, c)$ are functions of only the previous period's state $z_{-1}$, but $(d, b^u)$ can depend on the current state $z$ as well as $z_{-1}$. Future money holdings are denoted $m^g_{+1}$ if the household has restricted trades in the current period and $m^n_{+1}$ if the household has unrestricted trades. The household takes as given other households' decisions, aggregate variables and bond prices $(S, S^u)$.

The representative household solves the following problem:[9]

$$(PH) \quad v(m, b, z_{-1}) = \max_{(a,\ell,q,x,c)(z_{-1})} \left[g\mathcal{U}^g + (1-g)\mathcal{U}^n\right],$$

where, for $i \in \{g, n\}$, $\mathcal{U}^i$ is defined as:

$$\mathcal{U}^i \equiv u(c^i(z_{-1})) - \alpha\sigma(1-\sigma)\psi(Q^i) + \beta \int \max_{(d,b^u)(z,z_{-1})} v(m^i_{+1}, b_{+1}, z)\Phi(dz, z_{-1}).$$

The constraints of the problem are as follows:
(i) the constraints in the goods market, (2) – (4), and

$$c^i(z_{-1}) = \alpha\sigma(1-\sigma)q^i(z_{-1}), \quad i = n, g; \tag{5}$$

[9]To establish existence, uniqueness and differentiability of the value function, one can use the standard procedure in Stokey and Lucas with Prescott (1989). First, it is easy to show that the constraints in the problem form a closed, bound and convex subset in a finite Euclidean space. Given the assumptions on $u$ and $\psi$, the Theorem of the Maximum implies that the maximum in $(PH)$ exists and that the maximizer is unique. Second, since other households' decisions $(X^n, Q^n, X^g, Q^g)$ and the aggregate variables are taken as given in $(PH)$, the household's own marginal values of the assets, $\omega^m$ and $\omega^b$, do not appear directly on the right-hand side of $(PH)$. The mapping defined by the right-hand side satisfies Blackwell's sufficient conditions for contraction mapping, and so there exists a unique value function satisfying the Bellman equation in $(PH)$. Finally, differentiability of the value function follows from Theorem 4.11 in Stokey and Lucas with Prescott (p85).

(ii) the constraints in the bond market: $b^u(z) \geq 0$ and

$$S(z, z_{-1})\gamma d(z, z_{-1}) \leq [1 - a(z_{-1})]\, m + S^u(z, z_{-1})\, \{[1 - \ell(z_{-1})]\, b - b^u(z, z_{-1})\}\,; \tag{6}$$

(iii) the laws of motion of asset holdings:

$$b_{+1} = d(z, z_{-1}), \tag{7}$$

$$\begin{aligned} m^i_{+1} = \ & \tfrac{1}{\gamma}\, \{m - S(z, z_{-1})\gamma d(z, z_{-1}) + S^u(z, z_{-1})\, [(1 - \ell(z_{-1}))\, b - b^u(z, z_{-1})] \\ & + \alpha\sigma(1 - \sigma)\left[X^i - x^i(z_{-1})\right] + [\ell(z_{-1})b + b^u(z, z_{-1})] + L_{+1}\}\,, \ i = g, n. \end{aligned} \tag{8}$$

(iv) and other constraints: $0 \leq a(z_{-1}) \leq 1$ and $0 \leq \ell(z_{-1}) \leq 1$.

The objective function in the above problem contains two groups of terms, one for the case where the household's members are located in restricted matches and the other for the case in unrestricted matches.[10] The outer maximization in $(PH)$ determines the choices $(a, \ell, q, x, c)$, which are made before the realization of the shock $z$. The maximization in $\mathcal{U}^i$ determines the choices $(d, b^u)$, which maximize the future value function for each realization of $z$.

The constraints in (i) and (iv), and the law of motion of unmatured bonds, (7), are self-explanatory. In (ii), there are two constraints in the bond market. First, the household cannot hold a negative amount of unmatured bonds. Second, as (6) requires, the household must finance the purchase of new bonds by the assets it brings into the bond market. The last term in (6) is the receipt of money that the household obtains by selling some of the unmatured bonds it brings to the bond market. This amount is zero in a symmetric equilibrium.

To explain the law of motion of money, (8), recall that the household's money holding is measured at the time immediately after receiving monetary transfers and redeeming matured bonds (see Figure 1). Between two adjacent points of time of this measurement, money holdings can change as a result of the following transactions: purchasing newly issued bonds, selling unmatured bonds in the bond market, selling and buying goods, redeeming matured bonds and receiving the monetary transfer $L_{+1}$ next period. The terms following $m$ on the right-hand side of (8) list the net changes in money holdings from these transactions. Here, the factor $1/\gamma$ appears on the right-hand side because $m_{+1}$ is normalized by $M_{+1}$ while the money receipts in the current period are normalized by $M$.

In a symmetric equilibrium, all households hold the same portfolio of assets at the end of each period regardless of the matching shocks in the

[10]The implicit assumption here is that the goods in a restricted trade yield the same marginal utility as the goods in an unrestricted trade. For a relaxation of this assumption, see Shi (2005, 2008).

period. One reason for this result is that there is no communication between the markets. This implies that the bond-trading decisions, $(d, b^u)$, do not depend on the realization of the matching shock in the goods market. As a result, the amount of bonds carried into the next period will be independent of the matching shock. Another reason is that the representative household and its trading partners in the goods market experience the same matching shock. This feature implies $x^i = X^i$ in a symmetric equilibrium for $i = g, n$. Then, from (8) one can show that the amount of money holdings at the end of a period will also be independent of the matching shock, i.e., $m^g_{+1} = m^n_{+1}$. Thus, I can suppress the superscripts $(g, n)$ on $m$ and maintain a representative household over time.

To characterize optimal decisions, let $\rho(z, z_{-1})$ be the state-contingent Lagrangian multiplier of the constraint in the bond market, (6). Let $\lambda^n(z_{-1})\alpha\sigma(1-\sigma)(1-g)$ be the multiplier of the asset constraint in an unrestricted trade, (3), and $\lambda^g(z_{-1})\alpha\sigma(1-\sigma)g$ the multiplier of the asset constraint in a restricted trade, (4). The constants $\alpha\sigma(1-\sigma)(1-g)$ and $\alpha\sigma(1-\sigma)g$ in these multipliers are used to simplify various terms. For the moment, I suppress the dependence of $(a, \ell, q, x, c, \lambda)$ on $z_{-1}$ and of $(d, b^u, S, S^u, \rho)$ on $(z, z_{-1})$. The following conditions characterize the household's optimal choices.

(i) Quantities $q^n$ and $q^g$:

$$u'(c^i) = \left(\omega^m + \lambda^i\right) \frac{\psi'(q^i)}{\Omega^m}, \quad i = n, g. \tag{9}$$

(ii) Portfolio divisions $(a, \ell)$ and bond market decisions $(b^u, d)$:

$$\text{for } a\text{:} \quad \alpha\sigma\left[(1-g)\lambda^n + g\lambda^g\right] = \int \rho\Phi\left(dz, z_{-1}\right); \tag{10}$$

$$\text{for } \ell\text{:} \quad \alpha\sigma(1-g)\lambda^n + \omega^m = \int \left(\rho + \frac{\beta}{\gamma} v_{m_{+1}}\right) S^u \Phi(dz, z_{-1}); \tag{11}$$

$$\text{for } b^u\text{:} \quad \frac{\beta}{\gamma} v_{m_{+1}} = \left(\rho + \frac{\beta}{\gamma} v_{m_{+1}}\right) S^u; \tag{12}$$

$$\text{for } d\text{:} \quad \frac{\beta}{\gamma} v_{b_{+1}} = \left(\rho + \frac{\beta}{\gamma} v_{m_{+1}}\right) S. \tag{13}$$

In each of these conditions, the variable attains the lowest value in its domain if the equality is replaced by "<", and the highest value if ">", where $a, \ell \in [0, 1]$ and $d, b^u \in [0, \infty)$.

(iii) The envelope conditions for $m$ and $b$:

$$v_m = \omega^m + a\alpha\sigma\left[(1-g)\lambda^n + g\lambda^g\right] + (1-a)\int \rho\Phi\left(dz, z_{-1}\right); \tag{14}$$

$$v_b = \ell\left[\omega^m + \alpha\sigma(1-g)\lambda^n\right] + (1-\ell)\int\left(\rho + \frac{\beta}{\gamma}v_{m_{+1}}\right)S^u\Phi(dz, z_{-1}). \tag{15}$$

The condition (9) requires that the net gain to a buyer from asking for an additional amount of goods be zero. By getting an additional unit of good, the household's utility increases by $u'(c)$. The cost is to pay an additional amount $\psi'(q)/\Omega^m$ of assets in order to induce the seller to trade (see (2)). By giving an additional unit of asset, the buyer foregoes the discounted future value of the asset, $\omega^m$, and causes the asset constraint in the trade to be more binding. Thus, $(\omega^m + \lambda)$ is the shadow cost of each additional unit of asset to the buyer's household and the right-hand side of (9) is the cost of getting an additional unit of good from the seller.

In (ii), (10) says that for the household to allocate money to both the goods market and the bond market, money must generate the same expected liquidity services in the two markets. The liquidity services derive from the role of the asset in relaxing the trading constraints, as reflected by the shadow costs of the corresponding constraints.

The condition (11) is a similar requirement on the allocation of unmatured bonds between the two markets. If the household takes a unit of unmatured bond to the goods market, the bond can generate liquidity services $\alpha\sigma(1-g)\lambda^n$ by relieving the asset constraints and will have a future value $\frac{\beta}{\gamma}v_{m_{+1}}$ upon redemption. If the household instead takes the unit of unmatured bond to the bond market, the bond can be sold for $S^u$ units of money, which will generate liquidity services $\rho$ in the bond market and will have a future value $\frac{\beta}{\gamma}v_{m_{+1}}$. Because the household must choose $\ell$ before seeing the realization of $z$, it compares the expected values of allocating a marginal unit of unmatured bonds to the two markets. This comparison leads to (11).

The condition (12) specifies the optimal demand for unmatured bonds in the bond market. The value of keeping a unit of unmatured bond for future redemption is the discounted future value of one unit of money, $\frac{\beta}{\gamma}v_{m_{+1}}$. The value of selling a unit of unmatured bond for money is $\left(\rho + \frac{\beta}{\gamma}v_{m_{+1}}\right)S^u$, as explained above. For the choice $b^u$ to be interior, these two values must be equal to each other. The condition (13) is a similar requirement for the quantity of new bonds purchased, except that the price and future value of a new bond are different from those of an unmatured bond. Notice that (12) and (13) must hold for every realization of $z$.

Finally, the envelope conditions require the current value of each asset to be equal to the sum of the expected future value of the asset and the expected liquidity services generated by the asset in the current markets. Take the condition for money for example. The current value of money is $v_m$. The right-hand side of (14) consists of the expected future value of money, $\omega^m$, the liquidity services generated by money in the current bond market, $\rho$, and the liquidity services generated by money in the current goods market, $\lambda$. The liquidity services in the two markets are weighted by the division of money into the two markets.

### 2.5. Equilibrium Definition and Interest Rates

A (symmetric) monetary equilibrium consists of a value function $v$: $R_+ \times R_+ \times Z \to R$, portfolio division functions $a, \ell$: $Z \to [0,1]$, functions of trade quantities in matches $q^n, x^n, q^g, x^g$: $Z \to R_+$, consumption function $c$: $Z \to R_+$, bonds purchase functions $d, b^u$: $Z \times Z \to R_+$, bonds price functions $S, S^u$: $Z \times Z \to R_+$ such that the following requirements are met:

(i) Given other households' choices and $(m, b)$, the household's choices solve $(PH)$;

(ii) The choices are the same across households and, in particular, $m = 1$;

(iii) The bond market clears, i.e., $d(z, z_{-1}) = z$ and $b^u(z, z_{-1}) = [1 - \ell(z_{-1})]\, b$ for all $(z, z_{-1}) \in Z \times Z$;

(iv) $0 < \omega^m(z), \omega^b(z) < \infty$ for all $z \in Z$.

The requirements (i), (ii) and (iii) are self-explanatory. In part (iv), the restriction that the value of each asset be positive is necessary for a meaningful examination of the coexistence of money and bonds. The restriction that these values be bounded away from infinity is necessary for the first-order conditions to characterize optimal decisions.

Moreover, I restrict attention to equilibria in which money generates liquidity in the goods market in all states of the economy. That is, for all $z_{-1}$, at least one of $\lambda^n(z_{-1})$ and $\lambda^g(z_{-1})$ must be positive. Note that this restriction also implies $a(z_{-1}) > 0$ for all $z_{-1}$. However, it is not reasonable to assume that $\lambda^n$ and $\lambda^g$ are both positive for all realizations of the shock. As I will show later, $\lambda^n$ can be zero when the value of money is high.

By invoking equilibrium conditions, I can simplify the optimality conditions. First, because $d(z) = z \in (0, \infty)$ in equilibrium, the optimal condition for $d$ must hold as equality, as in (13). Second, because $m = 1$ and $b = d_{-1} = z_{-1}$ in equilibrium, I can shorten the notation for the shadow values of the assets as follows:

$$\mu^i(z_{-1}) \equiv v_i\left(1, z_{-1}, z_{-1}\right), \; i = m, b. \tag{16}$$

The expected value of $\mu$, defined in (1), can be expressed as $\omega^i = O(\mu^i)$, where

$$O(\mu^i)(z_{-1}) = \frac{\beta}{\gamma} \int \mu^i(z) \Phi\left(dz, z_{-1}\right), \; i = m, b. \tag{17}$$

Third, for all $S > 0$, the bond market clearing conditions imply $a < 1$. Under the restriction $a > 0$, then $0 < a < 1$, and the equality in (10) holds. The condition (14) can be simplified as

$$\mu^m(z_{-1}) = \omega^m(z_{-1}) + \alpha\sigma\left[(1-g)\lambda^n(z_{-1}) + g\lambda^g(z_{-1})\right]. \tag{18}$$

Now turn to the price of two-period bonds, $S$. If money yields liquidity in the bond market (i.e., if $\rho > 0$), then (6) binds and $S = (1-a)/(\gamma z)$. In this case, (13) implies $S < \mu^b(z)/\mu^m(z)$. If $\rho = 0$, then (6) does not bind. In this case, $S \leq (1-a)/(\gamma z)$, and (13) implies $S = \mu^b(z)/\mu^m(z)$. Combining the two cases, I express the two-period bond price as

$$S\left(z, z_{-1}\right) = \min\left\{\frac{1-a(z_{-1})}{\gamma z}, \frac{\mu^b(z)}{\mu^m(z)}\right\}. \tag{19}$$

I can also use (13) and (19) to obtain:

$$\rho\left(z, z_{-1}\right) = \frac{\beta}{\gamma} \max\left\{\frac{\gamma z}{1-a(z_{-1})}\mu^b(z) - \mu^m(z), 0\right\}. \tag{20}$$

Under the earlier restriction that at least one of $\lambda^g(z_{-1})$ and $\lambda^n(z_{-1})$ must be positive, the expected value of $\rho(z, z_{-1})$ over $z$ must be positive (see (10)). That is, the money constraint in the bond market binds "on average". However, the constraint does not bind with some realizations of $z$, as shown later in numerical examples.

The two-period nominal interest rate, defined as $r = S^{-1} - 1$, is

$$r(z, z_{-1}) = \max\left\{\frac{\gamma z}{1-a(z_{-1})} - 1, \frac{\mu^m(z)}{\mu^b(z)} - 1\right\}. \tag{21}$$

Notice that the past shock affects the current interest rate if and only if the following two conditions are satisfied. First, the money allocation in the current period, $a(z_{-1})$, depends on the past shock. Second, the maximum in (21) is equal to the first term inside the *max* operator, i.e., $\rho(z, z_{-1}) > 0$. These conditions are not always met, as I will illustrate in section 4. If both conditions are satisfied, then open market operations have persistent effects on interest rates.

The price of unmatured bonds in the bond market, $S^u$, depends on whether the household takes all unmatured bonds to the goods market. If

$\ell = 1$, the supply of and the demand for unmatured bonds in the bond market are both zero, in which case $S^u$ is indeterminate. If $\ell < 1$, the supply of unmatured bonds in the bond market is positive. In this case, the equality in (12) holds and so

$$S^u(z, z_{-1}) = \frac{\mu^m(z)}{\rho(z, z_{-1})\frac{\gamma}{\beta} + \mu^m(z)}. \tag{22}$$

Unmatured bonds are discounted if and only if $\rho(z, z_{-1}) > 0$.

Although the price of unmatured bonds may be indeterminate, the price of newly issued one-period bonds is determinate. If one-period bonds were issued, the price would be given by the right-hand side of (22) (regardless of whether $\ell < 1$). Denote this price as $S^I(z, z_{-1})$. Then,

$$\frac{S^I(z, z_{-1})}{S(z, z_{-1})} = \frac{\mu^m(z)}{\mu^b(z)}. \tag{23}$$

The ratio $\mu^m/\mu^b$ is the expected future discount on unmatured bonds. As shown later, $\mu^b(z) < \mu^m(z)$ in the equilibrium, because unmatured bonds are not perfect substitutes for money in the goods market. Thus, there is a deeper discount on two-period bonds than on one-period bonds.

## 3. CHARACTERIZATION AND EXISTENCE OF THE EQUILIBRIUM

One element that complicates the analysis of the equilibrium is that various money constraints may or may not be binding. In deterministic environments, one can avoid this complexity by restricting the parameter values and monetary policy so that the money constraints are either always binding or never binding. Such a restriction would be difficult to justify in a stochastic environment. For example, the money constraint in the bond market may bind for certain realizations of the shock but not for other realizations. This variation in the severity of the money constraint is a necessary implication of the liquidity effect of open market operations. To capture this effect, I need to allow for the possibility that the money constraints fail to bind for certain realizations of the shock.

### 3.1. Characterization

The equilibrium can be one of two cases, $0 \leq \ell < 1$ and $\ell = 1$. When $\ell = 1$, the household takes all unmatured bonds to the goods market and such bonds generate liquidity in a fraction $(1 - g)$ of trades, i.e., $\lambda^n > 0$. In the case $0 \leq \ell < 1$, unmatured bonds do not generate liquidity (at the margin) in the goods market, i.e., $\lambda^n = 0$, although some unmatured

bonds may still be used to buy goods.[11] Likewise, money generates liquidity services in unrestricted trades if and only if $\ell = 1$. In contrast to bonds, money also generates liquidity services in restricted trades if $\lambda^g > 0$.

To characterize the equilibrium, it is useful to express more explicitly the conditions under which the asset constraints bind. The condition $\lambda^n > 0$ is equivalent to $u'(c^n) > \psi'(q^n)$ (see (9) for $i = n$). Define $Q_0$ as the solution to the following equation:

$$u'\left(\alpha\sigma(1-\sigma)Q_0\right) = \psi'(Q_0). \tag{24}$$

Then, $\lambda^n > 0$ (and hence $\ell = 1$) iff $q^n < Q_0$. It is more convenient to express this condition in terms of the shadow value of money. Since $\ell = 1$ when $\lambda^n > 0$, then (3) and (2) imply that $\lambda^n > 0$ iff $0 < \omega^m < w_1$ where

$$w_1(a, z_{-1}) = \frac{1-\sigma}{a + z_{-1}}\psi(Q_0). \tag{25}$$

Similarly, the asset constraint binds in a restricted trade (i.e., $\lambda^g > 0$) iff $q^g < Q_0$, which can be rewritten as $0 < \omega^m < w_2$ where

$$w_2(a) = \frac{1-\sigma}{a}\psi(Q_0) > w_1(a, z_{-1}). \tag{26}$$

The equilibrium requires $\omega^m < w_2$, provided $\gamma > \beta$: If $\omega^m \geq w_2$, no asset would generate liquidity (at the margin) in the goods market, in which case $\mu^m = \omega^m$ and an equilibrium would exist only for $\gamma = \beta$. For $\omega^m < w_2$, the equilibrium falls into the following two cases.

**Case 1:** $0 < \omega^m < w_1$. In this case, $\lambda^n > 0$, $\lambda^g > 0$, and $\ell = 1$. Since the asset constraints (3) and (4) bind, the quantity of goods traded is $q^n = Q_1$ in an unrestricted trade and $q^g = Q_2$ $(< Q_1)$ in a restricted trade, where

$$Q_1(\omega^m; a, z_{-1}) = \psi^{-1}\left(\frac{a + z_{-1}}{1-\sigma}\omega^m\right), \tag{27}$$

$$Q_2(\omega^m; a) = \psi^{-1}\left(\frac{a\omega^m}{1-\sigma}\right). \tag{28}$$

The total amount of liquidity services that unmatured bonds generate is $\alpha\sigma(1-g)\lambda^n$. After substituting $\lambda^n$ from (9), I can express this amount as

[11] The proof for $\lambda^n = 0$ in this case is as follows. When $0 \leq \ell < 1$, the optimality condition for $\ell$ holds as "$\leq$"; That is, (11) holds as "$\leq$". Because $b^u = (1-\ell)b \in (0, \infty)$ when $0 \leq \ell < 1$, the equality in (12) holds, which leads to (22). Substituting (22) into the inequality form of (11) yields $\lambda^n \leq 0$. Thus, $\lambda^n = 0$.

$\omega^m(z_{-1})F^n$, where

$$F^n(\omega^m; a, z_{-1}) = \alpha\sigma(1-g)\left[\frac{u'\left(\alpha\sigma(1-\sigma)Q_1(\omega^m; a, z_{-1})\right)}{\psi'(Q_1(\omega^m; a, z_{-1}))} - 1\right]. \quad (29)$$

Similarly, the total amount of liquidity services that money generates is $\alpha\sigma(1-g)\lambda^n + \alpha\sigma g\lambda^g$, which can expressed as $\omega^m(z_{-1})(F^n + F^g)$ where $F^g$ is defined as follows:

$$F^g(\omega^m; a) = \alpha\sigma g\left[\frac{u'(\alpha\sigma(1-\sigma)Q_2(\omega^m; a))}{\psi'(Q_2(\omega^m; a))} - 1\right]. \quad (30)$$

**Case 2:** $w_1 \leq \omega^m < w_2$. In this case, $\lambda^n = 0 < \lambda^g$. Since $\lambda^n = 0$, the quantity of goods traded in an unrestricted trade is $q^n = Q_0$. Since $\lambda^g > 0$, the quantity of goods traded in a restricted trade is $q^g = Q_2$. Bonds do not generate liquidity services in this case. In contrast, money generates liquidity services, the total amount of which is $\omega^m(z_{-1})F^g$.

In both cases, $\mu^m(z) > \mu^b(z)$ for all $z$ (see (32) and (33)). Thus, unmatured bonds are not perfect substitutes for money in the goods market. As (23) shows, this imperfect substitutability induces a deeper discount on two-period bonds than on one-period bonds. The fraction of unmatured bonds taken to the goods market is $\ell = 1$ if $\omega^m < w_1$ and $\ell \in [0, 1)$ if $\omega^m > w_1$.[12]

To unify the two cases, express the total amount of liquidity services generated by money as $\omega^m(z_{-1})F$, where

$$F(\omega^m; a, z_{-1}) = \begin{cases} F^g(\omega^m; a) + F^n(\omega^m; a, z_{-1}), & \text{if } 0 < \omega^m \leq w_1 \\ F^g(\omega^m; a), & \text{if } w_1 \leq \omega^m \leq w_2 \\ 0, & \text{if } \omega^m \geq w_2. \end{cases} \quad (31)$$

Using this amount to substitute for the term $[\alpha\sigma(1-g)\lambda^n + \alpha\sigma g\lambda^g]$, I can write (18) as:

$$\mu^m(z_{-1}) = \omega^m(z_{-1})\left[1 + F(\omega^m(z_{-1}); a(z_{-1}), z_{-1})\right]. \quad (32)$$

[12] Although the value of $\ell$ is indeterminate when $\omega^m > w_1$, this indeterminacy of $\ell$ has no effect on real variables, because unmatured bonds in this case do not generate liquidity at the margin. The indeterminacy does not affect the equilibrium value of $a$, either, and hence the bond price $S$ and the corresponding interest rate $r$ do not depend on such indeterminacy.

Since $\omega^m = O(\mu^m)$, this is a functional equation for $\mu^m$ for any given function $a(.)$. Similarly, (15) yields a functional equation for $\mu^b$ as follows:[13]

$$\mu^b(z_{-1}) = \begin{cases} \omega^m(z_{-1})\left[1 + F^n(\omega^m(z_{-1}); a(z_{-1}), z_{-1})\right], & \text{if } 0 < \omega^m \leq w_1 \\ \omega^m(z_{-1}), & \text{if } \omega^m \geq w_1. \end{cases} \tag{33}$$

To determine the function $a(.)$, I use (10) to eliminate $(\lambda^g, \lambda^n)$ in (18), substitute the definition of $\omega^m$, and use (20) to eliminate $\rho$. This produces the following functional equation for $a$:

$$a(z_{-1}) = 1 - \frac{\beta/\gamma}{\mu^m(z_{-1})} \int \max\left\{\gamma z \mu^b(z),\ [1 - a(z_{-1})]\,\mu^m(z)\right\} \Phi(dz, z_{-1}). \tag{34}$$

The procedure for determining the equilibrium has two steps. First, for any fixed function $a(.)$ that is continuous and bounded in the interior of $[0, 1]$, I solve the fixed point for $\mu^m$ from (32). Substitute the solution into $\omega^m = O(\mu^m)$ to get $\omega^m$ and into (33) to get $\mu^b$. Then, substitute $(\mu^m, \mu^b)$ into the right-hand side of (34) to obtain a new function, denoted as $\Gamma_a(z_{-1})$. Second, the equilibrium function $a(.)$ solves $a(z_{-1}) = \Gamma_a(z_{-1})$ for all $z_{-1} \in Z$. Once the functions $(\mu^m, \mu^b, \omega^m, a)$ are determined, I can recover the traded quantities of goods and consumption (output) through (27) and (28), the bond price $S$ through (19) and the nominal interest rate through (21).

### 3.2. Existence of the Equilibrium

Two features of the equilibrium complicate the proof of existence. First, the mapping defined by the right-hand side of (32) is not concave. Second, the function $\Gamma_a$ (as a function of $a$) is an implicit one because it involves the solution to another fixed point problem. To get around these problems, I impose additional assumptions in this section. The proof has some resemblance to the proof by Lucas (1990). However, the details necessarily differ for the following reasons. First, the goods market here is non-Walrasian, and so prices are determined bilaterally. Second, output is endogenous, rather than being given by endowments. Third, there are two types of trades in the goods market – the restricted trades and unrestricted trades – and so prices are different in these trades. All proofs for this subsection are collected in Appendix A.1.1.

Let me begin by defining the bounds on various functions. Let $a$ be bounded in $[a_L, a_H]$ and $\mu^m$ bounded in $[\frac{\gamma}{\beta}\omega_L, \frac{\gamma}{\beta}\omega_H]$, where

$$0 < a_L \leq a_H < 1,\ 0 < \omega_L \leq \omega_H < \infty. \tag{35}$$

[13]The derivation is straightforward when $\omega^m < w_1$ (i.e., when $\ell = 1$). When $\omega^m > w_1$, $0 \leq \ell < 1$ and $\lambda^n = 0$, as discussed above. Then $b^u = (1 - \ell)b \in (0, \infty)$, and so the equality in (12) holds. This equality and the fact $\lambda^n = 0$ reduce (15) to $\mu^b = \omega^m$.

Then, by (17), $\omega^m$ is bounded in $[\omega_L, \omega_H]$. There will be further restrictions imposed on these bounds later in Lemma 1 and Theorem 1. Restrict attention to $\mu^m(.) \in \mathcal{V}$ and $a(.) \in \mathcal{A}$, where $\mathcal{V}$ denotes the set of continuous functions defined on $Z$ whose values lie in $[\frac{\gamma}{\beta}\omega_L, \frac{\gamma}{\beta}\omega_H]$ and $\mathcal{A}$ the set of continuous functions defined on $Z$ whose values lie in $[a_L, a_H]$. Endow $\mathcal{V}$ and $\mathcal{A}$ with the supnorm, so that they are complete metric spaces. Note that $\mathcal{V}$ and $\mathcal{A}$ are sets of functions defined on $Z$ which is assumed to be a finite set.

In the first step of the procedure for determining an equilibrium, I pick an arbitrary $a \in \mathcal{A}$ and solve for $\mu^m$ from (32). Denote the right-hand side of (32) as $T(\omega^m; a, z_{-1})$ and define

$$TO(\mu^m; a, z_{-1}) = T(O(\mu^m); a, z_{-1}). \tag{36}$$

Then, (32) requires $\mu^m$ to be the fixed point of $TO$. I will find conditions under which $TO$ is a monotone contraction mapping from $\mathcal{V}$ to $\mathcal{V}$.

Assumption 1. *Denote the relative risk aversion as* $\delta(c) = -cu''(c)/u'(c)$. *Assume that (i)* $\delta(c) \leq 1$, *and (ii) the function* $[1-\delta(c)]\, u'(c) \Big/ \psi'\left(\frac{c}{\alpha\sigma(1-\sigma)}\right)$ *is decreasing in c.*

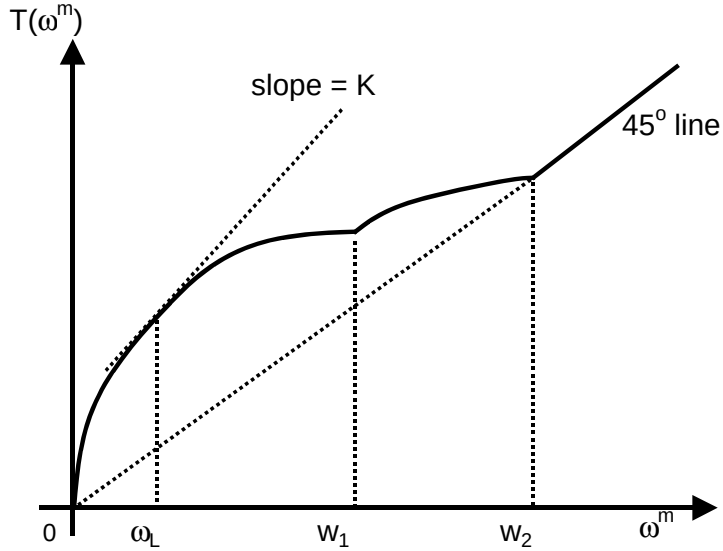
The unity upper bound on the relative risk aversion in part (i) simplifies the proofs by ensuring that $T(\omega^m; a, z_{-1})$ be increasing in $\omega^m$, but it is not necessary for existence. An equilibrium can exist even when the relative risk aversion exceeds one, as shown in the numerical examples in section 5.4. Also, the numerical examples will show that imposing this upper bound strengthens the quantitative predictions of the model. Part (ii) of the above assumption is necessary and sufficient for $T$ to be concave in $\omega^m$ in each of the three segments $(0, w_1)$, $(w_1, w_2)$, and $(w_2, \infty)$. It is satisfied if, for example, the utility function exhibits constant relative risk aversion. Figure 2 depicts the mapping $T$ on $\omega^m$.

I can select a constant $K$ that is sufficiently close to but greater than 1, and use $K$ to construct the lower bound $\omega_L$ (see Figure 2 for an illustration and Appendix A.1.1 for the formal construction). Then, the following lemma holds.

Lemma 1. *Let* $\varepsilon > 0$ *be a small number. Let* $K$ *be sufficiently close to but greater than* 1. *Given any function* $a(.) \in \mathcal{A}$, *the mapping* $TO$ *defined in (36) is a monotone contraction mapping from* $\mathcal{V}$ *to* $\mathcal{V}$ *if the following condition holds:*

$$\max\{K - 1 + \varepsilon, F(\omega_H, a_L, z_L)\} \leq \frac{\gamma}{\beta} - 1 \leq F(\omega_L, a_H, z_H). \tag{37}$$

**FIG. 2.**




*There is a non-empty set of parameter values that satisfy (37). Under this condition, TO has a unique fixed point* $\mu_a^m(.) \in \mathcal{V}$.

The subscript $a$ in the notation $\mu_a^m$ emphasizes the dependence of the fixed point for $\mu^m$ on the function $a$ that is arbitrarily chosen from $\mathcal{A}$ in this step of the analysis. Similarly, the expected future shadow value of money is $\omega_a^m(z_{-1}) = O(\mu_a^m)(z_{-1})$. The shadow value of unmatured bonds is obtained from (33) as $\mu_a^b(.)$. Clearly, $\omega_a^m(.)$ and $\mu_a^b(.)$ are continuous. Also, $\omega_a^m(z) \in [\omega_L, \omega_H]$ for all $z$.

In the second step of the procedure for determining an equilibrium, I solve the equilibrium function $a$. To do so, substitute $\mu_a^m$ and $\mu_a^b$ solved above into (34). I obtain $a(z_{-1}) = \Gamma_a(z_{-1})$, where

$$\Gamma_a(z_{-1}) \equiv 1 - \frac{\beta/\gamma}{\mu_a^m(z_{-1})} \int \max\left\{\gamma z \mu_a^b(z),\ [1 - a(z_{-1})]\, \mu_a^m(z)\right\} \Phi(dz, z_{-1}). \tag{38}$$

Treat $\Gamma$ as a mapping for $a$. Then, the equilibrium function $a$ is a fixed point of $\Gamma$. Once this fixed point is shown to exist, then the functions $\mu_a^m(.)$ and $\mu_a^b(.)$ will recover the shadow values of money and unmatured bonds. The following theorem summarizes the existence of the equilibrium.

THEOREM 1. *Maintain Assumption 1 and choose* $(\gamma, K, \omega_H)$ *to satisfy (37). Let* $a_H$ *be close to* 1. *There is a nonempty set of values of* $(z_H, a_L)$

*that satisfy the following condition:*

$$F(\omega_H, a_H, z_H) \geq \max\left\{\frac{\gamma z_H}{1-a_L} - 1, 0\right\}. \tag{39}$$

*Under (39),* $\Gamma$ *is a continuous mapping from* $\mathcal{A}$ *to* $\mathcal{A}$*. Thus, an equilibrium exists, which satisfies* $\mu^m(.) \in \mathcal{V}$*,* $a(.) \in \mathcal{A}$*, and* $\omega_L \leq \omega^m(z) \leq \omega_H$ *for all* $z \in Z$*.*

The restriction (39) requires $z_H$ to be sufficiently small. This restriction is necessary to ensure that the households allocate a positive fraction of money to the goods market. If the size of the open market operation were very large, instead, new bonds would be heavily discounted; given that the money growth rate is fixed, the households would allocate all money to the bond market to obtain the discount.

## 4. A SPECIAL CASE: INDEPENDENT SHOCKS

It is instructive to examine the special case where the shocks to bond sales are independent over time. With this special case, I illustrate the key differences between the current model and Lucas' (1990) in the effects of open market operations. I then explain how these differences depend on the two modelling elements that are absent in Lucas' model, namely, that bonds can circulate in the goods market and that output is endogenous.

The behavior of the equilibrium depends on whether unmatured bonds generate liquidity in the goods market. Consider first the case where unmatured bonds do not generate liquidity, i.e., where $\lambda^n = 0$ or equivalently, $\omega^m > w_1$. Then, the shadow values of assets, $(\mu^m, \mu^b)$, and the fraction, $a$, are numbers that are independent of the shocks. To verify this result, suppose that $(\mu^m, \mu^b, \omega^m, a)$ are all constants. Since $\lambda^n = 0$ in this case, $F = F^g$. Also, $\omega^m = O(\mu^m) = \frac{\beta}{\gamma}\mu^m$. Then, (32) becomes $F^g(\omega^m; a) = \frac{\gamma}{\beta} - 1$. Substituting $F^g$ from (30), this equation solves for the quantity of goods in a restricted trade, which is a constant. The quantity of goods in an unrestricted trade is also constant, given by $Q_0$. Moreover, $\mu^b = \omega^m$ by (33), and so $\mu^b = \frac{\beta}{\gamma}\mu^m$. With $(\mu^b, \mu^m)$, (34) becomes:

$$1 - a = \frac{\beta}{\gamma}\int max\{\beta z, 1-a\}\Phi(dz). \tag{40}$$

This equation solves for the constant $a$. Thus, $(\mu^m, \mu^b, \omega^m, a)$ are indeed constants in this case.

Equation (40) is essentially the same equation as in Lucas (1990) for the case of independent shocks. Consequently, this case of the current model

generates the liquidity effect that lasts for only one period, as in Lucas' model. To see this, substitute the above results for $(\mu^b, \mu^m, a)$ into (21):

$$r(z) = \max\left\{\frac{\gamma z}{1-a} - 1, \frac{\gamma}{\beta} - 1\right\}.$$

A high realization of the current shock reduces the current interest rate when $z < (1-a)/\beta$. This liquidity effect is not persistent, because future interest rates are independent of the current shock. The liquidity effect does affect real activities, either. Moreover, the additional discount on two-period bonds relative to one-period bonds is independent of the shocks, since it is equal to the constant $(\gamma/\beta - 1)$ (see (23)).

Continue the examination of the economy with independent shocks but now consider the case where unmatured bonds generate liquidity services, i.e., where $0 < \omega^m < w_1$. This case of the equilibrium behaves very differently from Lucas' model. In particular, $\mu^m$ and $a$ are no longer constants. Because the asset constraint binds in an unrestricted trade, the quantity of goods in such a trade depends on the amount of unmatured bonds, as well as the money stock. Since the amount of unmatured bonds in a period is equal to the quantity of new bonds issued in the previous period, the quantity of goods in an unrestricted match depends on the realization of the previous period's shock, $z_{-1}$ (see (27)). That is, the previous period's shock affects the amount of liquidity in the current goods market. As a result, current shadow values of the two assets are functions of the previous shock (see (32) and (33)). Since these asset values affect the allocation of money between the two markets, $a$ is a function of $z_{-1}$, even though the shocks are independent over time.

Now, the shocks to bond sales can generate persistent effects on the nominal interest rate, even when the shocks are independent. This can be seen from (21). Since the past shock affects the current allocation of money, the current interest rate depends on both the current shock and the past shock, provided that the money constraint binds in the bond market. Moreover, open market operations affect the relative value of unmatured bonds to money, and hence affect the term structure of interest rates.

The persistence of the liquidity effect relies on the assumptions that bonds can circulate in the goods market and that output is endogenous. Recall that Lucas (1990) assumes that individuals are not allowed to use bonds as payments in the goods market and that output is fixed as endowment. Endogenizing output in Lucas' model alone does not lead to persistent liquidity effects. Since there is a cash-in-advance constraint in Lucas' model, bonds do not generate liquidity in the goods market, no matter whether or not output is endogenous. This case is similar to the case $\lambda^n = 0$ analyzed above.

On the other hand, fixing output in the current model will also eliminate the persistence of the liquidity effect. To see this, suppose that every seller/producer is restricted to produce either 0 or a fixed amount $\bar{q} > 0$. Then, (2) implies $x^g = x^n = x \equiv \psi(\bar{q})/\Omega^m$. Since $x \leq am/(1-\sigma)$, the asset constraint does not bind in an unrestricted trade, provided $\ell > 0$ (see (3)). Again, $\lambda^n = 0$. In this case, with independent shocks, $a$ is a constant solving (40), and past shocks do not affect the current interest rate.

## 5. NUMERICAL EXAMPLES

Let me return to the general case where the shocks can be dependent. Because the equilibrium function $a$ is a fixed point of an implicit mapping $\Gamma$, it is difficult to check whether the solution is monotone. Likewise, it is difficult to check whether consumption is a monotonic function of the past shock. To study equilibrium properties, I turn to numerical examples.

### 5.1. Parameterization and Notation

Assume the following forms of utility and cost:

$$u(c) = u_0 \frac{c^{1-\delta} - 1}{1-\delta}, \quad \psi(q) = \psi_0 q^{\Psi}.$$

Let the shock $z$ have two realizations, $z_1$ and $z_2$, with $z_2 > z_1$. Refer to $z_2$ as the high shock and $z_1$ as the low shock. The transition probability from $z_i$ to $z_{i'}$ $(i' \neq i)$ is $1-\theta$, where $i, i' = 1, 2$, and the probability of staying at $z_i$ is $\theta$. Consider the following parameter values as the baseline:

| | |
|---|---|
| preference: | $\delta = 0.5$, $u_0 = 4$, $\psi_0 = 1$, $\Psi = 2$, $\beta = 0.995$; |
| goods market: | $\alpha = 1$, $\sigma = 0.5$, $g = 0.2$; |
| monetary policy: | $z_1 = 0.02$, $z_2 = 0.08$, $\gamma = 1.005$. |

The value of $g$ matches the size of the government relative to the economy, using the interpretation that the legal restriction in the goods market is imposed in trades between private households and the government.[14] The values of $(\beta, z_1, z_2)$ are the ones chosen by Lucas (1990). With the particular value of $\beta$, I can interpret the length of a period as one month and the interest rate $r$ as the bi-monthly interest rate. Also following Lucas, I explore a large range of values of $\theta$: $0.01, 0.1, 0.3, 0.5, 0.7, 0.9$ and $0.99$. I will also analyze the sensitivity of the results to other parameters in section 5.4.

[14]Notice that $g$ is not the fraction of goods purchased with money. Because money is used in the current model to buy both restricted goods and unrestricted goods, the fraction of goods purchased with money is much larger than the value of $g$.

The bounds on the variables are set at $a_L = 0.90$, $a_H = 0.98$, $\kappa = 1$, $\omega_L = 0.543$ and $\omega_H = 3.375$. These bounds satisfy all the conditions in Theorem 1. Moreover, the equilibrium lies in the region $\omega^m \in (0, w_1)$. That is, unmatured bonds generate liquidity in the goods market and the household takes all unmatured bonds to the goods market. This is the case for a large range of parameter values.

To display the results, let me add a subscript $i$ to variables that depend only on the previous period's shock $z_i$, where $i = 1, 2$. Add subscripts $ji$ to variables that depend on both the current shock $z_j$ and the previous period's shock $z_i$, where $i, j = 1, 2$. To aggregate consumption of the goods over the two types of trades, denote the price of goods in a restricted trade, normalized by the money stock, as $p^g$ and the normalized price in an unrestricted trade as $p^n$. Aggregate real consumption (output) is defined as follows:

$$c_i = \alpha\sigma(1-\sigma)\left[\frac{gp^g(z_i)q^g(z_i) + (1-g)p^n(z_i)q^n(z_i)}{gp^g(z_i) + (1-g)p^n(z_i)}\right].$$

Notice that current output depends only on the shock in the previous period, but not on the current shock. This is because consumption in the current period is purchased with the assets that are allocated to the goods market before the current shock is realized. Then, following the convention in asset pricing models, I can define the (ex ante) real interest rate between the current and the next period period as

$$\text{Ereal}_i = \left[\beta\mathbb{E}\frac{u'(c_j)}{u'(c_i)}\right]^{-1} - 1, \quad i = 1, 2, \tag{41}$$

where the expectation is taken over the current shock $z_j$, conditional on the past shock $z_i$.

The term structure is represented by the percentage difference between the yield to newly issued two-period bonds, $S^{-1/2}$, and the yield to one-period bonds, $1/S^I$. Letting $r^I$ be the one-period interest rate corresponding to $S^I$ and using (23), I can write this difference as:

$$term_{ji} = \left(\frac{\mu^m(z_j)/\mu^b(z_j)}{1 + r^I(z_j, z_i)}\right)^{1/2} - 1. \tag{42}$$

Table 1 describes equilibrium properties of the fraction of money taken to the bond market $(1-a)$, nominal and real interest rates, consumption, and the term structure of nominal interest rates. The mean, the standard deviation, and serial autocorrelations are calculated using the unique invariant measure $prob(z_i) = 1/2$ for $i = 1, 2$.

### 5.2. Results Similar to Lucas' Model

There are three important similarities between the results in Table 1 and those reported by Lucas (1990). First, interest rates change significantly with the persistence of the shock when the current shock is high. Also, interest rates have a large (unconditional) standard deviation. However, the mean of interest rates does not vary significantly with the persistence of the shock, even if the degree of persistence varies between 0.1 and 0.9. Thus, if one is interested only in the mean of interest rates, one can ignore the persistence and simply examine the case of independent shocks (i.e., $\theta = 0.5$).

Second, the fraction of money allocated to the bond market is insensitive to the previous period's shock. As in Lucas' model, this insensitivity is surprising especially when the shocks are negatively dependent. With negatively dependent shocks, a high shock in the previous period implies that the amount of bond sales is likely to be low in the current period and the bond price likely to be high. Since the discount on bonds will be small, there is not much need to allocate more money to the bond market to take advantage of the discount on new bonds. Thus, when the shocks are negatively correlated, one would expect that the household would reduce $(1 - a)$ significantly upon observing a high shock in the previous period. This does not happen in the numerical examples.

The insensitivity of money allocation to shocks is more puzzling here than in Lucas' model, because the goods market here provides an additional reason for the household to adjust the money allocation. In particular, a high past shock increases the amount of assets used in an unrestricted trade relative to the assets in a restricted trade. This widens the gap between the quantities of goods obtained in the two types of trades, and hence increases the variation in a household's consumption. To smooth consumption between the two types of trades, the household should increase the fraction of money allocated to the goods market, so as to maintain a stable ratio of assets used in an unrestricted trade relative to a restricted trade. Despite this additional motivation for changing $a$, the negative response of $(1 - a)$ to the past shock is not significant. Even when $\theta = 0.1$, an increase of $z_{-1}$ from $z_1$ to $z_2$ reduces $(1 - a)$ from 7.87% to 7.70%. This reduction is small in comparison with the variation in the shock.

Third, the insensitivity of the money allocation leads to a strong liquidity effect in the bond market. Interest rates are much higher when the current shock is high than when the current shock is low; that is, $r_{2i}$ is much higher than $r_{1i}$ for $i = 1, 2$. This is because the insensitive money allocation forces the bond price to fall in order to absorb the higher supply of new bonds. Notice that $r_{11} = r_{12}$; that is, the two-period interest rate is independent of past shocks when the current shock is low. This is because there is more money than what is needed in the bond market when the amount of

**TABLE 1.**

Table 1. Simulation results under a constant money growth rate

| | $\theta$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.01 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.99 |
| $1-a_1$ (%) | 7.879 | 7.867 | 7.828 | 7.761 | 7.615 | 7.044 | 3.952 |
| $1-a_2$ (%) | 6.224 | 7.698 | 7.892 | 7.930 | 7.941 | 7.926 | 7.903 |
| $r_{11}$ (%) | 0.474 | 0.451 | 0.445 | 0.439 | 0.428 | 0.389 | 0.327 |
| $r_{21}$ (%) | 2.038 | 2.201 | 2.705 | 3.593 | 5.580 | 14.133 | 103.46 |
| $r_{12}$ (%) | 0.474 | 0.451 | 0.445 | 0.439 | 0.428 | 0.389 | 0.327 |
| $r_{22}$ (%) | 29.178 | 4.443 | 1.877 | 1.384 | 1.248 | 1.441 | 1.732 |
| $\mathbb{E}(r)$ (%) | 1.392 | 1.438 | 1.451 | 1.464 | 1.488 | 1.549 | 1.538 |
| $StD(r)$ (%) | 2.119 | 1.096 | 1.041 | 1.289 | 1.759 | 2.932 | 7.259 |
| $corr(r, r_{-1})$ | −0.129 | −0.483 | −0.535 | −0.341 | −0.166 | −0.029 | 0.004 |
| $corr(r, r_{-2})$ | 0.126 | 0.387 | 0.214 | 0 | −0.067 | −0.023 | 0.004 |
| $corr(r, r_{-3})$ | −0.124 | −0.309 | −0.086 | 0 | −0.027 | −0.018 | 0.004 |
| $c_1$ | 0.616 | 0.617 | 0.617 | 0.617 | 0.618 | 0.619 | 0.622 |
| $c_2$ | 0.628 | 0.627 | 0.626 | 0.626 | 0.626 | 0.625 | 0.622 |
| $\mathbb{E}(c)$ | 0.622 | 0.622 | 0.622 | 0.622 | 0.622 | 0.622 | 0.622 |
| $StD(c)$ | 0.006 | 0.005 | 0.005 | 0.005 | 0.004 | 0.003 | 0.000 |
| $\text{Ereal}_1$ (%) | 1.450 | 1.231 | 1.042 | 0.869 | 0.702 | 0.545 | 0.503 |
| $\text{Ereal}_2$ (%) | −0.436 | −0.221 | −0.035 | 0.136 | 0.303 | 0.460 | 0.502 |
| $corr(c, r)$ | −0.298 | −0.537 | −0.554 | −0.429 | −0.276 | −0.073 | 0.025 |
| $corr(c_{+1}, r)$ | 0.433 | 0.901 | 0.966 | 0.795 | 0.603 | 0.396 | 0.167 |
| $corr(c_{+2}, r)$ | −0.424 | −0.721 | −0.386 | 0 | 0.241 | 0.317 | 0.163 |
| $corr(c_{+3}, r)$ | 0.416 | 0.577 | 0.155 | 0 | 0.096 | 0.253 | 0.160 |
| $\text{term}_{11}$ (%) | 0.237 | 0.225 | 0.222 | 0.219 | 0.214 | 0.194 | 0.163 |
| $\text{term}_{21}$ (%) | −0.720 | −0.687 | −0.902 | −1.299 | −2.181 | −5.782 | −29.391 |
| $\text{term}_{12}$ (%) | 0.237 | 0.225 | 0.222 | 0.219 | 0.214 | 0.194 | 0.163 |
| $\text{term}_{22}$ (%) | −11.764 | −1.758 | −0.500 | −0.230 | −0.111 | −0.061 | −0.144 |

single subscript $i(=1,2)$: conditional on the previous period's shock $z_i$;
double subscript $ji(i,j=1,2)$: conditional on past shock $z_i$ and current shock $z_j$.

bond sales is low in the current period. In this case, the cash-in-advance constraint in the bond market does not bind, the one-period interest rate is zero (see (22)), and the two-period interest rate is driven entirely by the imperfect substitution between unmatured bonds and money in the goods market in the next period.

### 5.3. New Results

The model generates several results that are absent in Lucas (1990). I will describe them for the case of independent shocks, since the contrasts with Lucas' model are the sharpest in this case.

First, open market operations have a real effect – A high shock in the previous period increases current real output. The standard deviation of output is about 0.7% of the mean. This real effect arises because (unmatured) bonds generate liquidity in the goods market. A high shock in the previous period increases the stock of unmatured bonds in the current goods market. This allows a buyer to purchase a larger quantity of goods in an unrestricted trade than if the previous period's shock was low, i.e., $q_2^n > q_1^n$. The presence of a larger quantity of nominal assets in the goods market also pushes up the price level and reduces the quantity of goods purchased in a restricted trade, i.e., $q_2^g < q_1^g$. In the numerical examples, the increase in $q^n$ dominates the decrease in $q^g$, and so aggregate output rises.[15] This positive effect of a tightening open market operation on output depends on the unrealistic assumption that open market operations do not affect money growth. When open market operations affect money growth, as they typically do in practice, a tightening open market operation is likely to reduce aggregate output, as will be illustrated in section 6.

As a result of the effect on consumption, the past shock also affects the real interest rate. When the shock was high in the previous period, current consumption is high and so the real interest rate is low in the current period. On the other hand, the real interest rate is high in the current period when the shock was low in the previous period. The difference in the real interest rate between the two states of the past shock is about 70 basis points, which is sizable. However, because of the liquidity effect, this difference is smaller than the one in the nominal interest rate between the two states.

Second, a high past shock reduces the current nominal interest rate when the current shock is high. In contrast, past (independent) shocks in Lucas' model do not affect the current interest rate. To explain this new effect, recall that a high past shock increases the amount of unmatured bonds circulating in the current goods market and increases the price level. The higher price level reduces real values of both money and unmatured bonds. However, because the increased amount of unmatured bonds increases liquidity in unrestricted trades, the real value of unmatured bonds ($\mu^b$) falls by less than does the real value of money ($\mu^m$). (In fact, $\mu^b$ in the

[15] Also, real output (consumption) is serially correlated (not reported in Table 1). The coefficient of correlation between current consumption and $k$-period past consumption is equal to $(2\theta - 1)^k$. Thus, positively correlated shocks induce positive autocorrelations in consumption.

examples barely changes at all with past shocks.) Thus, the relative value of unmatured bonds to money increases, which induces the households to allocate more money to purchase new bonds. When the current shock is high, the additional money in the bond market pushes up the bond price and depresses the current interest rate. When the current shock is low, the additional money in the bond market does not affect the current interest rate, as discussed above.

Third, the above effects of past shocks on current activities induce the following correlations: (i) Contemporaneous output and interest rates are negatively correlated; (ii) Interest rates in two adjacent periods are negatively correlated; (iii) Future output is positively correlated with the current interest rate.[16] These correlations arise because a shock in the previous period increases current output, increases the interest rate in the previous period, and reduces the current interest rate. Although the result (i) is realistic, the results (ii) and (iii) are not realistic. I will examine a natural variation of the model in section 6 that will eliminate these unrealistic features.

Fourth, the term structure of interest rates responds to open market operations. The yield curve is negatively sloped when the current shock is high and positively sloped when the current shock is low. In light of (42), this negative response of the yield curve to the current shock is not surprising. For example, a high current shock increases the one-period interest rate; at the same time, it reduces the expected future discount on unmatured bonds ($\mu^m/\mu^b$) by generating liquidity in next period's goods market. Both effects reduce the slope of the yield curve.

Moreover, the slope of the yield curve can depend on the previous period's shock: When the current shock is high, a high past shock makes the yield curve less negatively sloped. To explain this result, recall that a high past shock increases the money allocation to the bond market. This allocation reduces current interest rates when the current shock is high. However, the expected future discount on unmatured bonds depends only on the current shock, not on past shocks. Thus, by (42), the yield curve becomes less negatively sloped.[17] Clearly, the role of unmatured bonds in the goods market is important for this dependence of the yield curve on

[16] The formulas for the correlations between $r$ and $c$ are as follows:

$$corr(r,y) = \tfrac{y_2-y_1}{2}\left[\theta(r_{22}-r_{11})+(1-\theta)(r_{12}-r_{21})\right],$$
$$corr(r,y_{+1}) = \tfrac{y_2-y_1}{2}\left[\theta(r_{22}-r_{11})+(1-\theta)(r_{21}-r_{12})\right].$$

Moreover, $corr(r,y_{-j}) = (2\theta-1)^j corr(r,y)$ and $corr(r,y_{+j}) = (2\theta-1)^{j-1} corr(r,y_{+1})$, for $j = 1, 2, \ldots$.

[17] Of course, when the current shock is low, the one-period interest rate is zero and unaffected by the money allocation, in which case the slope of the yield curve is independent of past shocks.

past shocks, because it is the reason why the households condition their money allocation on past shocks. In contrast, in Lucas' model, bonds play no role in the goods market regardless of the maturity, and so the yield curve is independent of past shocks when the shocks are independent.

Most of the above features with independent shocks continue to exist when shocks are dependent. However, there are a few changes. First, when shocks are highly negatively dependent (i.e., $\theta \leq 0.1$), a high past shock increases (rather than decreases) the current interest rate when the current shock is high. That is, $r_{22} > r_{21}$. This is because, given the high past shock and the negative serial dependence, the households anticipate bond sales to be low in the current period and so they allocate less money to the bond market. When the current bond sales turn out to be high, the interest rate will be high. Second, when the shocks are highly persistent (i.e., $\theta \geq 0.99$), the correlation between the current and one-period past interest rates becomes positive. This is not surprising because a permanent shock will generate a positive correlation between interest rates in all periods. Similarly, the contemporaneous correlation between output and interest rates becomes positive when the shocks are highly persistent.

### 5.4. Sensitivity Analysis

In this section, I examine the sensitivity of the results to the parameters, while maintaining the assumption that the money growth rate is constant. I only examine the case where shocks are independent. The parameters to be perturbed are the money growth rate ($\gamma$), the scope of the legal restriction ($g$), the relative risk aversion ($\delta$) and the variation in the shock. Table 2 reports the sensitivity results.

Overall, these perturbations do not change the main features of the baseline model. In particular, the allocation of money between the markets is still insensitive to the shock and there is a strong liquidity effect in the bond market as reflected by the large responses of interest rates to the shocks. The specific effects of each perturbation are summarized below.

First, an increase in the money growth rate increases the mean of interest rates and reduces the mean of real consumption (output). It also increases the standard deviations of interest rates and real output. By eliminating net money growth from the baseline case, the mean and standard deviation of interest rates fall by about a half, and the standard deviation in output falls by more than a half. Real output and the nominal interest rate are still negatively correlated with each other but the magnitude seems to first increase, and then decrease, with money growth.

Second, an increase in the scope of the legal restriction increases the mean of interest rates but affects the standard deviation of interest rates in a hump-shaped pattern. The mean of real consumption barely changes with the increase in the scope of the legal restriction, the standard deviation of

consumption decreases, and the negative correlation between consumption and interest rates weakens. Real consumption responds in this way because the wider coverage of the legal restriction reduces the liquidity effect of unmatured bonds in the current goods market and reduces the variation in the quantity of goods between a restricted trade and an unrestricted trade. Because consumption varies less and interest rates vary more between different states, the two variables become less correlated with each other when $g$ increases.

**TABLE 2.**

Table 2. Sensitivity results under a constant money growth rate

| | | $\gamma$ | | $g$ | | $\delta$ | | $z_1 = 0.001$ |
|---|---|---|---|---|---|---|---|---|
| | baseline | 1 | 1.05 | 0.01 | 0.5 | 0.05 | 2 | $z_2 = 0.099$ |
| $1 - a_1$ (%) | 7.761 | 7.864 | 7.323 | 7.729 | 7.789 | 7.758 | 7.770 | 9.595 |
| $1 - a_2$ (%) | 7.930 | 7.933 | 7.583 | 8.033 | 7.847 | 7.939 | 7.902 | 9.843 |
| $\mathbb{E}(r)$ (%) | 1.464 | 0.783 | 7.056 | 1.044 | 1.823 | 1.429 | 1.578 | 1.361 |
| $StD(r)$ (%) | 1.289 | 0.598 | 5.850 | 1.725 | 1.058 | 1.319 | 1.197 | 1.382 |
| $corr(r, r_{-1})$ | −0.341 | −0.316 | −0.163 | −0.337 | −0.175 | −0.346 | −0.311 | −0.352 |
| $\mathbb{E}(c)$ | 0.622 | 0.626 | 0.588 | 0.622 | 0.622 | 0.507 | 0.789 | 0.622 |
| $StD(c)$ | 0.005 | 0.002 | 0.007 | 0.008 | 0.002 | 0.006 | 0.002 | 0.005 |
| $corr(c, r)$ | −0.429 | −0.371 | −0.168 | −0.571 | −0.181 | −0.447 | −0.361 | −0.473 |
| $corr(c_{+1}, r)$ | 0.795 | 0.851 | 0.971 | 0.589 | 0.967 | 0.775 | 0.860 | 0.744 |
| $\text{term}_{11}$ (%) | 0.219 | 0.137 | 0.685 | 0.014 | 0.399 | 0.203 | 0.274 | 0.166 |
| $\text{term}_{21}$ (%) | −1.299 | −0.577 | −5.301 | −1.927 | −0.774 | −1.353 | −1.125 | −1.446 |
| $\text{term}_{12}$ (%) | 0.219 | 0.137 | 0.685 | 0.014 | 0.399 | 0.203 | 0.274 | 0.166 |
| $\text{term}_{22}$ (%) | −0.230 | −0.141 | −3.605 | −0.015 | −0.404 | −0.212 | −0.288 | −0.180 |

Baseline: $\theta = 0.5$, $\gamma = 1.005$, $g = 0.2$, $\delta = 0.5$, $z_1 = 0.02$, $z_2 = 0.08$.
single subscript $i(= 1, 2)$: conditional on the previous period's shock $z_i$;
double subscript $ji$ $(i, j = 1, 2)$: conditional on past shock $z_i$ and current shock $z_j$.

In the limit $g \to 1$, every trade requires cash. Then, the real effect of open market operations will diminish to zero as in Lucas' (1990) model. In the opposite limit $g \to 0$ (and with independent shocks), the real effect will also diminish to zero because there will be no dispersion in the quantities across trades. It is then surprising to see from Table 2 that, even in the case with $g = 0.01$, open market operations affect output significantly.

Third, an increase in the relative risk aversion increases the mean and reduces the standard deviation of interest rates and output. It also reduces the magnitude of the (negative) correlation between output and interest rates. The important features of the model do not change much with the relative risk aversion. Even when the relative risk aversion is very small, e.g., when $\delta = 0.05$, the money allocation remains insensitive to the previous period's shock. Note that an equilibrium still exists for $\delta = 2$, a

value that does not satisfy Assumption 1.[18] Also, note that, when $\delta = 2$, the money allocation between the two markets is even more insensitive to the previous period's shock than in the baseline model. In this sense, by restricting the relative risk aversion in the baseline model to be not greater than 1, I have strengthened the model's predictions.

Fourth, an increase in the mean-preserving spread in the shock reduces the mean and increases the variation in interest rates. It also increases the variation in output, without affecting the mean of output much, and strengthens the negative correlation between output and the interest rate.

Finally, the above perturbations affect the magnitude but not the sign of the slope of the yield curve. In particular, when $g$ becomes very small, the yield curve becomes very flat.

## 6. ALLOWING THE MONEY GROWTH RATE TO VARY

I have so far maintained the assumption that open market operations do not affect money growth. Although this assumption allowed me to compare the model's predictions with Lucas', it produced some counterfactual results: A tight operation increases real output, and future output is positively correlated with current interest rates when the shocks are positively correlated. It is important to check whether these counterfactual results are caused by the unrealistic assumption on money growth. To do so, I now assume that monetary transfers in each period are a fixed fraction of the money stock. That is, $L_{+1} = \tau M$, where $\tau$ is constant.

Denote $\gamma = M_{+1}/M$. By (8) and the requirements (ii) and (iii) in the equilibrium definition (see section 2.5), I have:

$$\gamma = \gamma(z_{-1}) \equiv a(z_{-1}) + z_{-1} + \tau. \tag{43}$$

Thus, the past shock $z_{-1}$ affects the growth rate of the aggregate money stock between the current period and the next period. Notice that this growth rate does not depend on the current shock $z$, because the current shock affects neither the amount of money that the household spends in the current bond market nor the amount of bonds that the household will redeem at the beginning of the next period. Also, the autocorrelation in money growth is the same as that in the shocks. Thus, money growth rates in two adjacent periods are positively correlated if and only if $\theta > 1/2$.[19]

[18] For $\delta = 2$, the function $T(\omega^m; a, z_{-1})$ is first decreasing and then increasing in $\omega^m$ as $\omega^m$ increases. To ensure that $T$ is increasing in $\omega^m$, the lower bound on $\omega^m$ is chosen to be sufficiently large.

[19] To see this, denote $\gamma_i = \gamma(z_i)$ and $\mathbb{E}\gamma = (\gamma_1 + \gamma_2)/2$. Then, $\mathbb{E}(\gamma_{+1}|\gamma = \gamma_i) = 2(1-\theta)\mathbb{E}\gamma + (2\theta - 1)\gamma_i$. The serial correlation in money growth is equal to $(2\theta - 1)$.

Once $\gamma$ is replaced with $\gamma(z_{-1})$, all equilibrium conditions in section 2 continue to hold and an equilibrium can still be characterized as in section 3.1. For the existence of an equilibrium, the conditions in Theorem 1 need be modified to incorporate the fact that now $\gamma$ is not a constant. These modifications are straightforward and hence are omitted here.

For the numerical exercise, let the parameter values (except $\gamma$) be the same as specified at the beginning of section 5. Let the fraction of monetary transfers be equal to the average of the two realizations of $z$, i.e., $\tau = 0.05$. The results are reported in Table 3.

**TABLE 3.**

Table 3. Simulation results under varying money growth rates

| | | $\theta$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | baseline | 0.01 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.99 |
| $1-a_1$ (%) | 7.761 | 7.356 | 7.537 | 7.544 | 7.477 | 7.339 | 6.748 | 2.026 |
| $1-a_2$ (%) | 7.930 | 2.152 | 6.462 | 7.629 | 7.841 | 7.900 | 7.873 | 7.826 |
| $r_{11}$ (%) | 0.439 | 1.436 | 0.906 | 0.723 | 0.640 | 0.545 | 0.444 | 3.627 |
| $r_{21}$ (%) | 3.593 | 8.364 | 5.577 | 5.472 | 6.482 | 8.640 | 18.845 | 314.51 |
| $r_{12}$ (%) | 0.439 | 3.029 | 0.906 | 0.723 | 0.640 | 0.545 | 0.444 | 1.183 |
| $r_{22}$ (%) | 1.384 | 312.12 | 31.891 | 10.489 | 7.298 | 6.434 | 6.825 | 7.505 |
| $\mathbb{E}(r)$ (%) | 1.464 | 7.208 | 4.557 | 3.850 | 3.765 | 3.821 | 4.236 | 7.089 |
| $StD(r)$ (%) | 1.289 | 21.779 | 6.670 | 3.524 | 3.138 | 3.352 | 4.569 | 21.880 |
| $corr(r, r_{-1})$ | −0.341 | −0.010 | −0.045 | −0.050 | 0.065 | 0.247 | 0.354 | 0.003 |
| $c_1$ | 0.617 | 0.578 | 0.599 | 0.606 | 0.609 | 0.613 | 0.617 | 0.588 |
| $c_2$ | 0.626 | 0.596 | 0.607 | 0.609 | 0.606 | 0.602 | 0.594 | 0.587 |
| $\mathbb{E}(c)$ | 0.622 | 0.587 | 0.603 | 0.607 | 0.608 | 0.608 | 0.605 | 0.587 |
| $StD(c)$ | 0.005 | 0.009 | 0.004 | 0.001 | 0.001 | 0.005 | 0.001 | 0.001 |
| $corr(c, r)$ | −0.429 | −0.050 | −0.083 | −0.056 | −0.065 | −0.253 | −0.427 | −0.016 |
| $corr(c_{+1}, r)$ | 0.795 | 0.193 | 0.547 | 0.887 | −0.996 | −0.977 | −0.830 | −0.159 |
| $corr(c_{+2}, r)$ | 0 | −0.189 | −0.438 | −0.355 | 0 | −0.391 | −0.664 | −0.156 |
| $corr(c_{+3}, r)$ | 0 | 0.185 | 0.350 | 0.142 | 0 | −0.156 | −0.531 | −0.153 |
| $\text{term}_{11}$ (%) | 0.219 | 0.715 | 0.452 | 0.351 | 0.320 | 0.272 | 0.222 | −0.603 |
| $\text{term}_{21}$ (%) | −1.299 | −2.624 | −1.610 | −1.596 | −2.012 | −2.885 | −6.936 | −50.076 |
| $\text{term}_{12}$ (%) | 0.219 | −0.067 | 0.452 | 0.351 | 0.320 | 0.272 | 0.222 | 0.590 |
| $\text{term}_{22}$ (%) | −0.230 | −50.067 | −11.970 | −3.956 | −2.385 | −1.884 | −1.840 | −1.969 |
| $\gamma_1 - 1$ (%) | 0.500 | −0.356 | −0.537 | −0.544 | −0.477 | −0.339 | −0.252 | 4.974 |
| $\gamma_2 - 1$ (%) | 0.500 | 10.848 | 6.538 | 5.371 | 5.159 | 5.100 | 5.127 | 5.174 |

Baseline: $\theta = 0.5$, $\gamma = 1.005$ (fixed), $g = 0.2$, $\delta = 0.5$, $z_1 = 0.02$, $z_2 = 0.08$.
single subscript $i$ $(= 1, 2)$: conditional on the previous period's shock $z_i$;
double subscript $ji$ $(i, j = 1, 2)$: conditional on past shock $z_i$ and current shock $z_j$.

Similar to the baseline model, this model generates large increases in interest rates when there is a positive shock to bond sales. As before, this strong liquidity effect arises because the allocation of money between the bond market and the goods market is insensitive to the shock. Moreover, the term structure of interest rates exhibits the same pattern of dependence on the shocks as in the baseline model.

By allowing the shocks to affect money growth, the model eliminates the unrealistic features mentioned above in the baseline model. For brevity, let me focus on the case $\theta = 0.7$. The model implies the following features. First, nominal interest rates are positively correlated. Second, a tightening open market operation reduces output. Third, current output and future output in three consecutive periods are negatively correlated with the current interest rate. These features suggest that a tightening open market operation increases interest rates and reduces output persistently.

To explain these persistent effects, let me first examine how a past shock affects the money growth rate. When the shock in the previous period was high, there will be a large amount of redemption at the beginning of the next period. Since the money allocation is not sensitive, this amount of redemption will be the dominant force determining the money growth rate between the current period and the next period (see (43)). Thus, a high past shock increases the money growth rate between the current period and the next period.

Now I can explain the persistent effects of the open market operation. Suppose that there was a high shock to bond sales in the pervious period and that the shocks are positively correlated. Then, the money growth rate between the current period and the next period rises. Expected inflation increases, which raises the current interest rate. Because the high past shock increased the interest rate in the previous period through the liquidity effect in the bond market, interest rates are positively correlated in the two adjacent periods. Also, expected inflation reduces the real value of money in the current period and reduces current output. Moreover, because the shocks are positively correlated, inflation is expected to be high in future periods. This will reduce output in future periods. Therefore, future output and current output are all negatively correlated with the interest rate.

The persistent effects of the shocks on interest rates and output are realistic (see Christiano et al., 1999). Although they rely on the positive effect of a past shock on future money growth, they do not require the money growth rate to be negatively correlated. On the contrary, in the case examined above (with $\theta = 0.7$), the autocorrelation of money growth is equal to 0.4, which is a realistic number.

## 7. CONCLUSION

In this paper I combine a decentralized goods market and a centralized bond market to analyze the liquidity effect of open market operations. The bond market features limited participation, while the goods market features bilateral matches. In a fraction of matches, a legal restriction prevents buyers from using bonds to pay for goods. In such a restricted trade, the buyer faces a money constraint. In an unrestricted trade, the buyer can use both money and unmatured bonds to buy goods, and so unmatured bonds can provide liquidity. A shock to bond sales in this economy has two distinct liquidity effects. One is the immediate liquidity effect in the bond market and the other is a liquidity effect in the goods market starting one period later.

The liquidity effect in the bond market arises for the same reason as in Lucas (1990). That is, there is limited participation in the bond market, and the households' money allocation between the markets is insensitive to past shocks even when shocks are highly persistent. As a result, the bond price and the interest rate absorb most of the shock to bond sales.

The liquidity effect in the goods market is new and it occurs with one-period delay. To describe this additional liquidity effect, suppose that the money growth rate is fixed irrespective of open market operations. Then, a high shock to bond sales in the previous period increases the amount of unmatured bonds circulating in the current goods market, relaxes the asset constraints in unrestricted trades, and increases the quantity of goods traded in an unrestricted trade. Although inflation also rises to reduce the quantity of goods traded in a restricted trade, the increase in the quantity of goods traded in unrestricted trades can dominate. In this case, aggregate output rises with a high past shock. This delayed liquidity effect also changes the amount of money that households allocate to the bond market, affects the current interest rate, and hence makes interest rates serially correlated.

When money growth is not fixed but positively correlated, a positive shock to bonds sale reduces real output. In addition, the model generates the following features: (i) Current output is negatively correlated with the current nominal interest rate; (ii) Output in three consecutive future periods, starting from the next period, is negatively correlated with the current nominal interest rate; (iii) Nominal interest rates in two adjacent periods are positively correlated; (iv) There is a non-trivial term structure of interest rates and the slope of the yield curve depends on both past and current shocks. These features indicate that open market operations have persistent effects on output and interest rates when there is search in the goods market.

Let me remark on the delayed liquidity effect in the goods market. First, the effect arises only when prices and quantities of goods in an unrestricted trade respond to the shock in the previous period differently from those in a restricted trade. For this reason, decentralized exchanges in the goods market are important for the delayed liquidity effect. If all agents could move assets between trades, then a high past shock would push up prices to such a level that would eliminate most of the output response. Second, the duration of the delayed liquidity effect increases with the length of maturity of the bonds that are used in open market operations.[20] Third, the real effect is different from that in the literature of limited participation. This literature imposes a separate cash-in-advance constraint on firms' payments on investment or wages, and open market operations affect output by changing the loanable funds for such payments.

Another important message that this paper tries to convey is that it is tractable to use a monetary model with a strong microfoundation to analyze the stochastic relationships between aggregate variables that have been the focus of traditional macro models. Often, search monetary models are criticized as being internationally consistent but difficult to be integrated with the rest of macroeconomic theory (e.g., Kiyotaki and Moore, 2001). The model described in this paper is no more difficult than many of the models in the literature of limited participation (see Christiano et al., 1999, for references). Thus, I hope that this paper has eliminated a major road block to the integration of the microfoundation of monetary theory into mainstream macroeconomics.

It is useful to extend the model by relaxing the following assumptions that I have retained from Lucas' model. First, the shock to bond sales is the only shock in the economy and, in particular, there are no shock to money demand or to the production technology. Second, there is no element (other than the one-period separation between markets) to delay the transmission of shocks from the bond market to the goods market. Third, there is no capital accumulation that can prolong the effects of monetary shocks. Despite the absence of these elements, the model is still able to generate persistent liquidity effects. Nevertheless, one may want to introduce these realistic elements to examine the monetary propagation mechanism. For example, money demand shocks can be modelled as stochastic changes in the scope of the legal restriction, $g$. Finally, there may be a need to model explicitly how financial institutions create inside money and to endogenize the legal restriction.

[20] If the government attaches repurchase agreements to bond sales, then the duration of the liquidity effect of bonds in the goods market will be reduced.

# APPENDIX A

## A.1. PROOFS FOR SECTION 3.2

### A.1.1. Proof of Lemma 3.1

To begin, I construct an upper bound on $T_\omega \equiv \partial T(\omega^m; a, z_{-1})/\partial\omega^m$. This upper bound is necessary for $TO$ to satisfy the contraction mapping requirement. It is easy to verify that $T_\omega > 0$ under Assumption 1. Also, $T$ is concave in $\omega^m$ in each of the three segments, $(0, w_1)$, $(w_1, w_2)$ and $(w_2, \infty)$ (see Figure 2). Thus, $T_\omega \leq \max\{T_\omega(w_1+; a, z_{-1}), 1\}$ for all $\omega^m \geq w_1$. Also, because $T_\omega(w_1\text{-}; a, z_{-1}) < T_\omega(w_1+; a, z_{-1})$, there exists $w_3 < w_1$ such that for all $\omega^m \geq w_3$, $T_\omega \leq \max\{T_\omega(w_1+; a, z_{-1}), 1\}$. Under (ii) of Assumption 1, $T_\omega(w_1+; a, z_{-1})$ decreases in $a$ and increases in $z_{-1}$, after the dependence of $w_1$ on $(a, z_{-1})$ is taken into account. Setting $a = a_L$, $z_{-1} = z_H$ and $w_1 = w_1(a_L, z_H)$, I have $T_\omega(w_1+; a, z_{-1}) \leq \bar{T}_\omega$ for all $(a, z_{-1})$, where

$$\bar{T}_\omega \equiv 1 - \alpha\sigma g + \frac{\alpha\sigma g}{\Psi} \frac{[1 - \delta(\widehat{c})]\, u'(\widehat{c})}{\psi'\left(\frac{\widehat{c}}{\alpha\sigma(1-\sigma)}\right)},$$

$$\widehat{c} = \alpha\sigma(1-\sigma)\psi^{-1}\left(\frac{a_L\psi(Q_0)}{a_L + z_H}\right).$$

Choose the upper bound on $T_\omega$ as

$$K = \kappa \max\{\bar{T}_\omega, 1\}, \text{ where } 1 \leq \kappa < \infty.$$

The upper bound $K$ leads to a lower bound on $\omega^m$. Let $\omega_0(a, z_{-1})$ $(< w_1)$ solve $T_\omega(\omega_0; a, z_{-1}) = K$. Because $T_\omega(\omega; a, z_{-1})$ is decreasing in $(\omega, a, z_{-1})$, $\omega_0$ is decreasing in $(a, z_{-1})$. The lower bound of $\omega^m$ is then defined as $\omega_L = \omega_0(a_L, z_L)$. Clearly, $\omega_L$ is smaller if a larger $\kappa$ is chosen (see Figure 2), and $\omega_L > 0$ for all finite $\kappa$. Also, for all $\omega^m \geq \omega_L$, $0 < T_\omega \leq K$.

Next, I show that $TO$ maps from $\mathcal{V}$ to $\mathcal{V}$. For any $\mu^m \in \mathcal{V}$, $O(\mu^m)$ is continuous because $\Phi$ has the Feller property. Since $a(.) \in \mathcal{A}$ is continuous, $TO(\mu^m)$ is continuous. Because $\mu^m \geq \frac{\gamma}{\beta}\omega_L$, then $O(\mu^m) \geq \omega_L$. Hence,

$$TO(\mu^m) \geq \omega_L\left[1 + F\left(\omega_L; a, z_{-1}\right)\right] \geq \omega_L\left[1 + F\left(\omega_L; a_H, z_H\right)\right].$$

The first inequality comes from the fact that $T$ is an increasing function of $\omega^m$ and the second inequality from the fact that, for given $\omega^m$, $F(\omega^m; a, z_{-1})$ is a decreasing function of $(a, z_{-1})$. Similarly, $O(\mu^m) \leq \omega_H$ and

$$TO(\mu^m) \leq \omega_H\left[1 + F(\omega_H; a, z_{-1})\right] \leq \omega_H\left[1 + F(\omega_H; a_L, z_L)\right].$$

Thus, $TO(\mu^m) \in \left[\frac{\gamma}{\beta}\omega_L, \frac{\gamma}{\beta}\omega_H\right]$ if

$$F(\omega_H, a_L, z_L) \geq \frac{\gamma}{\beta} - 1 \geq F(\omega_H; a_L, z_L).$$

This is part of (37) in the lemma.

Moreover, $TO$ is a contraction mapping under the supnorm. To see this, take any $\mu', \mu'' \in \mathcal{V}$. Let $\omega' = O(\mu')$ and $\omega'' = O(\mu'')$. Then, $\omega', \omega'' \geq \omega_L$ and

$$|\omega' - \omega''| = |O(\mu') - O(\mu'')| \leq \frac{\beta}{\gamma} \|\mu' - \mu''\|.$$

Because $T$ is concave in each of its segments and because $T_\omega$ is bounded above by $K$ for all $\omega^m \geq \omega_L$, I have:

$$|T(\omega') - T(\omega'')| \leq K |O(\mu') - O(\mu'')| \leq \frac{\beta}{\gamma} K \|\mu' - \mu''\|.$$

Thus, $\|TO(\mu') - TO(\mu'')\| \leq \frac{\beta}{\gamma} K \|\mu' - \mu''\|$. The mapping $TO$ is a contraction if $\gamma/\beta \geq K + \varepsilon$, where $\varepsilon > 0$. This completes the condition (37) in the lemma.

Because $TO$: $\mathcal{V} \to \mathcal{V}$ is a contraction mapping under (37), and $\mathcal{V}$ (with the supnorm) is a complete metric space, $TO$ has a unique fixed point $\mu_a^m \in \mathcal{V}$.

Finally, there is a nonempty set of parameter values that satisfy (37). To show this, note that $F(\omega_L, a_H, z_H) > 0$ by construction. By choosing $\kappa$ sufficiently close to 1, I can ensure that $K$ is sufficiently close to one, and so $K + \varepsilon < F(\omega_L, a_H, z_H) + 1$. Then, there are values of $\gamma$ ($> \beta$) that satisfy $K + \varepsilon \leq \gamma/\beta \leq F(\omega_L, a_H, z_H) + 1$. Also, because $F(\omega; a, z) = 0$ when $\omega$ is large, I can choose a large value for $\omega_H$ to ensure $F(\omega_H, a_L, z_L) + 1 \leq \gamma/\beta$. Clearly, these conditions require $\gamma > \beta$ and $\omega_H > \omega_L$. **QED**

### A.1.2. Proof of Theorem 1

To prove Theorem 1, I first show that $\Gamma$ defined by (38) maps $\mathcal{A}$ into $\mathcal{A}$. The following lemma gives the sufficient conditions for this result.

LEMMA 2. *Given any $a \in \mathcal{A}$, $\Gamma_a \in \mathcal{A}$ if $a_H$ is close to one and if (39) is satisfied.*

*Proof.* Since $\mu_a^m \geq \frac{\gamma}{\beta}\omega_L > 0$, $(\mu_a^m, \mu_a^b)$ are continuous, and $\Phi$ has the Feller property, then $\Gamma_a(.)$ defined by (38) is continuous. To show $\Gamma_a \in \mathcal{A}$, it suffices to show $\Gamma_a(z) \in [a_L, a_H]$ for all $z \in Z$. Notice that the right-hand side of (38) is increasing in $a(z_{-1})$ for given $(\mu_a^m, \mu_a^b)$. Since

$a(z_{-1}) \in [a_L, a_H]$, the sufficient conditions for $\Gamma_a(z_{-1}) \in [a_L, a_H]$ are:

$$\mu_a^m(z_{-1}) \leq \frac{\beta}{\gamma} \int \max\left\{\frac{\gamma z}{1-a_H}\mu_a^b(z_{-1}),\ \mu_a^m(z)\right\}\Phi(dz, z_{-1}), \tag{A.1}$$

$$\mu_a^m(z_{-1}) \geq \frac{\beta}{\gamma} \int \max\left\{\frac{\gamma z}{1-a_L}\mu_a^b(z),\ \mu_a^m(z)\right\}\Phi(dz, z_{-1}). \tag{A.2}$$

The first condition is satisfied when $a_H$ is close to 1. For the second condition, note that $\mu_a^b(z) \leq \mu_a^m(z)$ for all $z$ (see (32) and (33)), and so

$$\begin{aligned}\text{RHS(A.2)} &\leq max\left\{\tfrac{\gamma z_H}{1-a_L},\ 1\right\}\tfrac{\beta}{\gamma}\textstyle\int \mu_a^m(z)\Phi(dz, z_{-1}) \\ &= \omega_a^m(z_{-1}) max\left\{\tfrac{\gamma z_H}{1-a_L},\ 1\right\}.\end{aligned}$$

Also, because $F(\omega^m; a, z_{-1})$ is decreasing in $(\omega^m; a, z_{-1})$, then

$$\mu_a^m(z_{-1}) = \omega_a^m(z_{-1})\left[1 + F(\omega_a^m(z_{-1}); a(z_{-1}), z_{-1})\right] \geq \omega_a^m(z_{-1})\left[1 + F(\omega_H; a_H, z_H\right].$$

Therefore, (39) is a sufficient condition for (A.2). This completes the proof of Lemma 2. ▮

It is possible to satisfy (39) by choosing $z_H$ and $a_L$ sufficiently close to 0. Thus, there is a nonempty set of parameter values in which $\Gamma$ maps $\mathcal{A}$ into $\mathcal{A}$.

Next, I show that $\Gamma : \mathcal{A} \to \mathcal{A}$ is continuous. Treat $\mu_a^m$, $\mu_a^b$ and $\omega_a^m$ as functions of $a$. I show that $(\mu_a^m, \mu_a^b, \omega_a^m)$ are continuous in $a$ in the supnorm. Once this is done, it is clear from (38) that $\Gamma$ is continuous in $a$ in the supnorm. Because the proofs for $(\mu_a^m, \mu_a^b, \omega_a^m)$ to be continuous in $a$ are similar, I describe only the proof for $\mu_a^m$. For the latter, I need to show that for any $\varepsilon > 0$, there exists $\Delta > 0$ such that $\left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| < \varepsilon$ whenever $\|a_2 - a_1\| < \Delta$, where the norm is the supnorm. Let $\varepsilon > 0$ be an arbitrary number. Define

$$B(\omega^m, z_{-1}) = \max_{a, \widehat{a} \in \mathcal{A}} \left|\frac{F(\omega^m, a(z_{-1}), z_{-1}) - F(\omega^m, \widehat{a}(z_{-1}), z_{-1})}{\widehat{a}(z_{-1}) - a(z_{-1})}\right|,$$

where $F$ is defined in (31). Since $F$ is decreasing in $a$, then $B > 0$. Also, because the intervals $[a_L, a_H], [\omega_L, \omega_H]$, and $[z_L, z_H]$ are bounded away from zero and bounded above, it can be verified that $B(\omega^m, z_{-1}) < \infty$. For any $a_1, a_2 \in \mathcal{A}$, if $\|a_2 - a_1\| < \Delta$, then

$$\begin{aligned}&|F(\omega^m, a_2(z_{-1}), z_{-1}) - F(\omega^m, a_1(z_{-1}), z_{-1})| \\ &\leq B(\omega^m, z_{-1})\,|a_2(z_{-1}) - a_1(z_{-1})| \leq B(\omega^m, z_{-1})\,\|a_2 - a_1\| < B(\omega^m, z_{-1})\Delta.\end{aligned}$$

Because $T(\omega^m, a, z_{-1}) = \omega^m (1 + F)$ and $\omega^m \leq \omega_H$, then

$$\begin{aligned}&|T(\omega^m, a_2(z_{-1}), z_{-1}) - T(\omega^m, a_1(z_{-1}), z_{-1})| \\ &= \omega^m |F(\omega^m, a_2(z_{-1}), z_{-1}) - F(\omega^m, a_1(z_{-1}), z_{-1})| < \omega_H B(\omega^m, z_{-1})\Delta.\end{aligned}$$

Since $\mu_a^m = T(\omega_a^m, a(z_{-1}), z_{-1})$ and $\|TO(\mu') - TO(\mu'')\| \leq \frac{\beta}{\gamma} K \|\mu' - \mu''\|$, I get:

$$\begin{aligned} &\left|\mu_{a_2}^m (z_{-1}) - \mu_{a_1}^m (z_{-1})\right| \\ = &\left|T\left(\omega_{a_2}^m, a_2(z_{-1}), z_{-1}\right) - T(\omega_{a_1}^m, a_1(z_{-1}), z_{-1})\right| \\ \leq &\left|TO\left(\mu_{a_2}^m, a_2(z_{-1}), z_{-1}\right) - TO(\mu_{a_1}^m, a_2(z_{-1}), z_{-1})\right| \\ &\quad + \left|T\left(\omega_{a_1}^m, a_2(z_{-1}), z_{-1}\right) - T(\omega_{a_1}^m, a_1(z_{-1}), z_{-1})\right| \\ < &\ \frac{\beta}{\gamma} K \left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| + \omega_H B(\omega_{a_1}^m(z_{-1}), z_{-1})\Delta. \end{aligned}$$

Taking the maximum over $z_{-1}$ on both sides of the inequality yields

$$\left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| < \frac{\Delta}{1 - \frac{\beta}{\gamma} K} \omega_H \|B\|.$$

Let $\Delta = \varepsilon \left(1 - \frac{\beta}{\gamma} K\right) \Big/ [\omega_H \|B\|]$. Because $\gamma/\beta > K$, $\|B\| < \infty$ and $0 < \omega_H < \infty$, then $\Delta > 0$. For all $a_1, a_2 \in \mathcal{A}$ such that $\|a_2 - a_1\| < \Delta$, $\left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| < \varepsilon$. Therefore, $\Gamma : \mathcal{A} \to \mathcal{A}$ is continuous.

Finally, we can verify that $\mathcal{A}$ is compact and convex. Because $\Gamma$ is continuous and $\mathcal{A}$ contains functions defined on a finite set $Z$, Brouwer's fixed point theorem implies that $\Gamma$ has a fixed point in $\mathcal{A}$. This fixed point is the equilibrium function $a(.)$. This completes the proof of Theorem 1.

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