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Risk Aversion, Uncertainty, Unemployment Insurance Benefit and Duration of "Wait" Unemployment*

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Building on the tools developed for American call options in financial markets in Finance and the optimal timing of investment under uncertainty in Economics, this paper proposes a stylized equilibrium model to study the optimal time for a risk-averse unemployed individual, who receives an unemployment insurance benefit and may receive a recall from the old job, to exit from a waiting (and hence unemployment) state and start a new job. It is shown that as a result of the unemployed individual's exercising her optimal timing strategy, there is duration of so-called waiting and that this duration is affected by a number of economic primitives, prominent among which are the attitude of an unemployed individual toward risk and the uncertainty faced by this individual.

Key Words: Unemployment insurance; Income; Utility function; Brownian motions; Search; Waiting; Exit; Continuation region.
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1. INTRODUCTION

Duration of unemployment has been studied extensively in the search literature in Economics.[1] These studies stress the effect of search on the duration of unemployment.[2] In a standard search model, an unemployed individual faces a random wage offer from a new job in each period at a constant search cost. The individual can decide either to accept the offer and become employed, or to reject the offer and stay unemployed. The random wage offers follow a fixed independent probability distribution. In this model, the unemployed individual will continue to search until the wage offer exceeds her reservation wage. There are variants of this standard search model which take into account the possibility of recall in order to accommodate the well-documented fact cited for instance in Feldstein (1975) and Bradshaw and Scholl (1976) that, in the United States, a significant portion of the laid-off workers is subsequently recalled and rehired by their former employers.[3]

This paper takes a different tack from the extant literature in looking at the issue of duration of unemployment. That is, we take the perspective of an unemployed individual, who is risk averse and lives in a world characterized by ubiquitous uncertainty, and who receives unemployment insurance benefit and, for the sake of generality, may receive a recall from the old job. In particular this paper is concerned with the value of "waiting" after a preferred job has been located by an unemployed individual through a search activity in the labor market.[4] We stress that it is not the objective of this paper to provide an alternative explanation for the duration of unemployment to that given in the search literature in Economics. Instead, this paper analyzes the behavior of an unemployed individual during the waiting state after a successful completion of a search in the labor market and before the re-entry into the employment status. Specifically, in this paper, we model the behavior of a risk-averse unemployed individual who has successfully located a preferred job through a search activity in the labor market, and who faces an uncertain income stream associated with a new job. To this end we use an equilibrium framework provided in Wang and Wirjanto (2004), which judiciously exploit the tools developed in American call options in financial markets in Finance and the theory of

[1]See Stigler (1961), McCall (1970), Lucas, and Prescott (1974), Lippman and McCall (1976a,b), Mortensen (1977), Pissarides (1982), Albrecht and Axell (1984), Katz (1986), Sargent (1987), Hvidding (1978), and Katz and Meyer (1990a,b).

[2]For other models of unemployment duration, see Javanovic (1979) and Mortensen (1982).

[3]See Pissarides (1982), Katz (1986), and Katz and Meyer (1990a,b).

[4]These two factors, search and waiting, are related. In fact, there have been an attempt at an interpretation of unemployment due to search as a wait unemployment. Notably, Hvidding (1978) points out the possibility of an alternative "wait" interpretation.

the optimal timing of investment under uncertainty in Economics.[5] Then we examine the effect of various parameters on the optimal time for the individual to stop receiving unemployment insurance (or welfare) benefits and to begin receiving the income from the new job. The continuous-time model proposed in this paper, which is more general than Sargent's (1987) discrete-time model, allows for stochastic unemployment insurance payments, and provides a closed-form solution for the optimal timing problem of the unemployed individual. This closed-form expression is used to study explicitly the effect of individual's preferences and other parameters on the duration of the waiting. This paper shows that the optimal timing decision and the duration of waiting depend on the amount of unemployment insurance (or welfare) benefits received by the individual, uncertainty of the income from the new job, the individual's attitude towards risk, and the individual's time preference. It is hoped that this theoretical analysis adds to our understanding on how risk aversion on the part of an unemployed individual and how uncertainty faced by this individual affect the duration of "wait" unemployment on her part.

The remainder of this paper is organized as follows. Section 2 sets up the basic model. Section 3 provides some comparative statics results regarding the effect of exogenous variables, such as the individual's attitudes toward risk, the unemployment insurance benefits received by the individual etc., on the duration of waiting. Section 4 establishes for a more general setting the existence of the waiting phenomenon. Section 5 concludes the paper. Proofs are collected in appendices.

2. THE BASIC MODEL

Consider an unemployed individual who at time $t = 0$, after searching in the labor market, has located a preferred job. The individual is aware of a possibility that at any time she may be recalled by her former employer, but does not know the exact timing when this will occur.[6] The decision that the unemployed individual must make in this situation is whether to take the job immediately or to wait. In making this decision, the individual realizes that the decision is not once-and-for-all. Instead she can always revaluate her decision at a later time. Thus a decision to wait for the recall for now is, in effect, a decision on the optimal timing of taking the new job.

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[5] This approach builds on the analogy between the timing of investment and American call options in financial markets - see e.g. McDonald and Siegel (1986), Dixit (1992), Pindyck (1991), Dixit and Pindyck (1994) and the "real option" literature cited therein.

[6] Following the search literature, we incorporate the possibility of recall in our model. The results of the paper, however, do not depend on the existence of the possibility of recall.

Let the individual's utility function be given by $u(x)$. Let X_t be a diffusion process on a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, P)$, where Ω is the state space, $\mathcal{F}$ a σ–algebra of subsets of Ω representing events, $\{\mathcal{F}_t\}$ is a family of nondecreasing sub–σ–algebras of $\mathcal{F}$ representing the information available at time t, and P a probability measure. The process X_t describes the income that the individual receives in the waiting state, which consists of the unemployment insurance (or welfare) benefits before the recall, or the wage income received from the old job following the recall. The timing of the recall is described by a stopping time τ_c. We allow the time before the recall to be infinity in some states, in which case there will be, in effect, no recall. Let Y_t be a diffusion process on $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, P)$ which describes the wage income to be received from the new job.

Suppose that the individual decides to start the new job at time τ. Then her intertemporal utility is

$$\mathbb{E}\left\{\int_0^\tau e^{-\rho t}u(X_t)\ dt + \int_\tau^\infty e^{-\rho t}u(Y_t)\ dt\Big|\mathcal{F}_0\right\}, \tag{1}$$

where ρ denotes her time-preference parameter. In a world with ubiquitous uncertainty, it is reasonable to assume that information is gradually revealed over time and as events realize. Therefore, in choosing her timing strategy τ, the individual will take into consideration all of the information available to her at the time of decision. Making the timing strategy, τ, contingent on the available information, the individual's optimal timing decision in effect becomes one of finding a stopping time τ^* that solves the problem

$$\max_\tau \mathbb{E}\left\{\int_0^\tau e^{-\rho t}u(X_t)\ dt + \int_\tau^\infty e^{-\rho t}u(Y_t)\ dt\Big|\mathcal{F}_0\right\}, \tag{2}$$

where the maximum is taken over all of the stopping times on the filtered probability space. Let τ^* be a solution to the utility maximization problem. Then τ^* gives the optimal timing for starting the new job. The waiting state will cease to exist if either there is a recall or the individual decides to switch to the new job. Thus the unemployment spell is given by $\min\{\tau_c, \tau^*\}$. As in Pissarides (1982), Katz (1986), Katz and Meyer (1990a,b) we will treat the timing of the recall as exogenous.[7] This means that, to find the duration of the waiting state, $\min\{\tau_c, \tau^*\}$, we only need to solve for τ^*.

To be more concrete in the ensuing analysis, the individual's utility function is assumed to take a specific functional form

$$u(x) = x^\gamma/\gamma, \qquad 0 \neq \gamma \leq 1, \tag{3}$$

[7]The exogenous probability of the recall is $P(\{\tau_c < \infty\})$.

where γ is the relative risk-aversion parameter of the individual. The diffusion process X_t is assumed to evolve according to a geometric Brownian motion

$$dX_t = \mu_1 X_t dt + \sigma_1 X_t dB_t^1, \quad X_0 = x > 0, \tag{4}$$

where μ_1 is the growth rate parameter of the process X_t, σ_1 is the variance parameter, and B^1 is a standard Brownian motion. The process Y_t is assumed to be given by

$$dY_t = \mu_2 Y_t dt + \sigma_2 Y_t dB_t^2, \quad Y_0 = y > 0, \tag{5}$$

with the obvious notation and B^2 is a standard Brownian motion that is independent of B^1. It is assumed that $\rho > \mu_i\gamma + \sigma_i^2\gamma(\gamma-1)/2$ for $i = 1, 2$. Equations (4) and (5)imply that

$$X_t = x \exp\left\{\left(\mu_1 - \frac{1}{2}\sigma_1^2\right) t + \sigma_1 B_t^1\right\}$$

and

$$Y_t = y \exp\left\{\left(\mu_2 - \frac{1}{2}\sigma_2^2\right) t + \sigma_2 B_t^2\right\}.$$

The filtration $\{\mathcal{F}_t\}$ is taken to be the family of σ–algebras generated by B^1 and B^2 augmented in the usual manner.

Since both X_t and Y_t are time-homogeneous Markov processes, the state of the system, which represents the information available to the individual at time t, is simply given by the values of the processes at time t, (X_t, Y_t). Therefore the conditional expectation

$$\mathbb{E}\left\{\int_t^{t+\tau} e^{-\rho(s-t)} u(X_s)\ ds + \int_{t+\tau}^{\infty} e^{-\rho(s-t)} u(Y_s)\ ds \middle| \mathcal{F}_t\right\}. \tag{6}$$

can be written as

$$\mathbb{E}^{(X_t,Y_t)}\left\{\int_0^{\tau} e^{-\rho s} u(X_s)\ ds + \int_{\tau}^{\infty} e^{-\rho s} u(Y_s)\ ds\right\}. \tag{7}$$

In particular, at $t = 0$, we have $(X_0, Y_0) = (x, y)$ as the starting point of the system and the conditional expectation in (6) can be expressed as

$$\begin{aligned}
&\mathbb{E}\left\{\int_0^{\tau} e^{-\rho s} u(X_s)\ ds + \int_{\tau}^{\infty} e^{-\rho s} u(Y_s)\ ds \middle| \mathcal{F}_0\right\} \\
&= \mathbb{E}^{(x,y)}\left\{\int_0^{\tau} e^{-\rho s} u(X_s)\ ds + \int_{\tau}^{\infty} e^{-\rho s} u(Y_s)\ ds\right\}.
\end{aligned}$$

Some comments about the processes X_t and Y_t are in order. Ignoring the possibility of the recall, we might argue that the process X_t should be deterministic during the period of waiting and perhaps should vanish after some cut-off point, since unemployment insurance (or welfare) benefit schemes per period generally come in the form of a fixed amount of cash and may terminate at some point in time. This point of view, however, can be restrictive. First, it assumes away the effect of inflation uncertainty which renders the benefits from the unemployment insurance (or welfare) in real terms uncertain. Secondly, it implicitly assumes that an unemployed person is unable to use the income stream from the unemployment insurance (or welfare) benefits and the financial market to finance her consumption stream. Thirdly, in practice, the terminal date of the unemployment benefits is not at all clear-cut. For instance, when the unemployment insurance benefit runs out, the unemployed person can go on to the welfare program, which, in principle, may last indefinitely in some countries.

It is also worthwhile to emphasize that the process Y_t represents the individual's net income generated from the new job. The net income from the job is defined as the gross income from the job minus any costs due to training for new skills or relocation to a new place. The cost need not be a direct cost. Instead it is defined as the payment from a scheme that finances the direct cost. In other words we have embedded in the process Y_t the way in which the "investment" or "sunk" cost associated with the new job is financed.

By assuming Y_t to be a geometric Brownian motion, we have essentially assumed that wages co-move with the business cycle. More formally it follows from the expression of Y_t that

$$F_{Y_t}(t|y) \geq F_{Y_t}(t|y'), \qquad \text{for } y' \geq y,$$

which means that $F_{Y_t}(\cdot|y)$ is stochastically increasing in the state y. This is a continuous-time version of assumption 1 in Lippman and McCall (1976a).

In this paper we restrict the states of the system to be in the set $M \equiv [\underline{x}, \bar{x}] \times (0, \bar{y}]$, where $\underline{x}$ and $\bar{x}$ are the minimum possible payment from the unemployment compensation and the maximum wage payment from the old job, respectively, $\bar{y}$ is the maximum payment from the new job.[8] More specifically we consider the processes

$$X'_t \equiv X_{t \wedge \tau_{(\underline{x}, \bar{x})}} \tag{8}$$

and

$$Y'_t \equiv Y_{t \wedge \tau_{(0, \bar{y})}}, \tag{9}$$

[8]See the later part of this section for more discussions on the upper bound $\bar{y}$.

where $\tau_{(\underline{x},\bar{x})}$ and $\tau_{(0,\bar{y})}$ are the exit times from $(\underline{x},\bar{x})$ and $(0,\bar{y})$ respectively. Here $\underline{x}$ and $\bar{x}$ are absorbing boundaries for X_t', and $\bar{y}$ is an absorbing boundary for Y_t'.

Both X_t' and Y_t' are the processes of interest to us and, at the risk of abusing the notation, we will use the same notations X_t and Y_t to denote them respectively. Define two operators on the spaces of twice-continuously-differentiable functions in the intervals $[\underline{x},\bar{x}]$ and $(0,\bar{y}]$ respectively, by

$$\mathcal{A}_X = \mu_1 x\frac{\partial}{\partial x} + \frac{1}{2}\sigma_1^2 x^2\frac{\partial^2}{\partial x^2}, \quad \text{if } \underline{x} < x < \bar{x}, \text{ otherwise } \mathcal{A}_X = 0 \tag{10}$$

$$\mathcal{A}_Y = \mu_2 y\frac{\partial}{\partial y} + \frac{1}{2}\sigma_2^2 y^2\frac{\partial^2}{\partial y^2}. \quad \text{if } 0 < y < \bar{y}, \text{ otherwise } \mathcal{A}_Y = 0 \tag{11}$$

These are the generators of the processes of X_t and Y_t respectively. Now let W_X denote the intertemporal utility of the individual derived from the income process X_t, i.e.,

$$W_X(x) = \mathbb{E}^{(x,y)}\left\{\int_0^\infty e^{-\rho t}u(X_t)\ dt\right\}.$$

Let τ be a stopping time. Then

$$\mathbb{E}^{(x,y)}\left\{\int_0^\tau e^{-\rho t}u(X_t)\ dt\right\} = \mathbb{E}^{(x,y)}\left\{\int_0^\infty e^{-\rho t}u(X_t)\ dt - \int_\tau^\infty e^{-\rho t}u(X_t)\ dt\right\}.$$

By Lemma (A.1) in the appendix and the strong Markov property of process X_t,

$$\mathbb{E}^{(x,y)}\left\{\int_\tau^\infty e^{-\rho t}u(X_t)\ dt\right\} = \mathbb{E}^{(x,y)}\left\{e^{-\rho\tau}W_X(X_\tau)\right\},$$

and W_X satisfies

$$\mathcal{A}_X W_X - \rho W_X + u = 0, \qquad x \in (\underline{x},\bar{x}). \tag{12}$$

Thus

$$\mathbb{E}^{(x,y)}\left\{\int_0^\tau e^{-\rho t}u(X_t)\ dt\right\} = W_X(x) - \mathbb{E}^{(x,y)}\left\{e^{-\rho\tau}W_X(X_\tau)\right\}.$$

Similarly we have

$$\mathbb{E}^{(x,y)}\left\{\int_\tau^\infty e^{-\rho t}u(Y_t)\ dt\right\} = \mathbb{E}^{(x,y)}\left\{e^{-\rho\tau}W_Y(Y_\tau)\right\},$$

where W_Y satisfies

$$\mathcal{A}_Y W_Y - \rho W_Y + u = 0, \qquad y \in (0, \bar{y}). \tag{13}$$

Thus we have

$$\begin{aligned}
&\mathbb{E}^{(x,y)}\left\{\int_0^\tau e^{-\rho t} u(X_t)\ dt + \int_\tau^\infty e^{-\rho t} u(Y_t)\ dt\right\} \\
&= W_X(x) + \mathbb{E}^{(x,y)}\left\{e^{-\rho\tau}[W_Y(Y_\tau) - W_X(X_\tau)]\right\}.
\end{aligned}$$

So the individual's optimal timing decision problem is equivalent to finding a stopping time that solves

$$\sup_\tau \mathbb{E}^{(x,y)}\left\{e^{-\rho\tau}[W_Y(Y_\tau) - W_X(X_\tau)]\right\}, \tag{14}$$

where the supremum is taken over all $\{\mathcal{F}_t\}$–stopping times. Intuitively, in finding the optimal timing strategy, the individual evaluates the tradeoff between the utility, $e^{-\rho\tau} W_Y(Y_\tau)$, from the income earned from the new job, and the utility, $e^{-\rho\tau} W_X(X_\tau)$, from the income process X_t.

THEOREM 1. *The exit time τ_D is a solution to (14), where D, called a continuation region, is defined as follows. Let*

$$\begin{aligned}
a_1 &= \left[\frac{(\rho - \mu_2\gamma - \sigma_2^2\gamma(\gamma-1)/2)\beta_1}{(\rho - \mu_1\gamma - \sigma_1^2\gamma(\gamma-1)/2)(\beta_1 - \gamma)}\right]^{1/\gamma} \\
\beta_1 &= \frac{\sqrt{\left[\sigma_1^2\gamma - \left(\frac{\sigma_1^2}{2} - \mu_1\right) + \left(\frac{\sigma_2^2}{2} - \mu_2\right)\right]^2 + 2(\sigma_1^2 + \sigma_2^2)\left[\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma-1)\right]}}{\sigma_1^2 + \sigma_2^2} \\
&\quad + \frac{\sigma_1^2\gamma - \left(\frac{\sigma_1^2}{2} - \mu_1\right) + \left(\frac{\sigma_2^2}{2} - \mu_2\right)}{\sigma_1^2 + \sigma_2^2} \\
a_2 &= \left[\frac{\beta_2(\rho - \mu_2\gamma - \sigma_2^2\gamma(\gamma-1)/2)}{\rho(\beta_2 - \gamma)}\right]^{1/\gamma} \qquad a_3 = a_2 \\
\beta_2 &= \frac{1}{\sigma_2^2}\sqrt{\left(\frac{\sigma_2^2}{2} - \mu_2\right)^2 + 2\sigma_2^2\rho} + \frac{1}{\sigma_2^2}\left(\frac{\sigma_2^2}{2} - \mu_2\right) \qquad \beta_3 = \beta_2 \\
a_4 &= \left[\frac{\beta_4(\rho - \mu_1\gamma - \sigma_1^2\gamma(\gamma-1)/2)}{\rho(\beta_4 - \gamma)}\right]^{1/\gamma} \\
\beta_4 &= \frac{-1}{\sigma_1^2}\sqrt{\left(\frac{\sigma_1^2}{2} - \mu_1\right)^2 + 2\sigma_1^2\rho} + \frac{1}{\sigma_1^2}\left(\frac{\sigma_1^2}{2} - \mu_1\right).
\end{aligned}$$

Assume that $\beta_1 \neq \gamma$. Then $\beta_2 > \gamma$ and $\beta_4 < \gamma$.

(i)If $\bar{y} > \max\{a_1\bar{x}, a_3\bar{x}\}$, then the continuation region D is given by,

$$D = \left\{ (x,y) \in M : \left\{ \begin{array}{ll} y < a_1 x & \textit{if } \underline{x} < x < \bar{x}; \\ y < a_2 \underline{x} & \textit{if } x = \underline{x}; \\ y < a_3 \bar{x} & \textit{if } x = \bar{x}. \end{array} \right\} \right\}$$

(ii)If $\bar{y} \leq \max\{a_1\bar{x}, a_3\bar{x}\}$, then the continuation region D is given by,

$$D = \left\{ (x,y) \in M : \left\{ \begin{array}{ll} y < a_1 x & \textit{if } \underline{x} < x < \bar{x} \textit{ and } y < \bar{y}; \\ y < a_2 \underline{x} & \textit{if } x = \underline{x} \textit{ and } y < \bar{y}; \\ y < a_3 \bar{x} & \textit{if } x = \bar{x} \textit{ and } y < \bar{y}; \\ x > a_4 \bar{y} & \textit{if } y = \bar{y}. \end{array} \right\} \right\}$$

Furthermore letting

$$g^*(x,y) = \sup_{\tau} E^{(x,y)} \left\{ e^{-\rho\tau} [W_Y(Y_\tau) - W_X(X_\tau)] \right\}, \tag{15}$$

and $M = (0, \bar{x}] \times (0, \bar{y}]$, we have

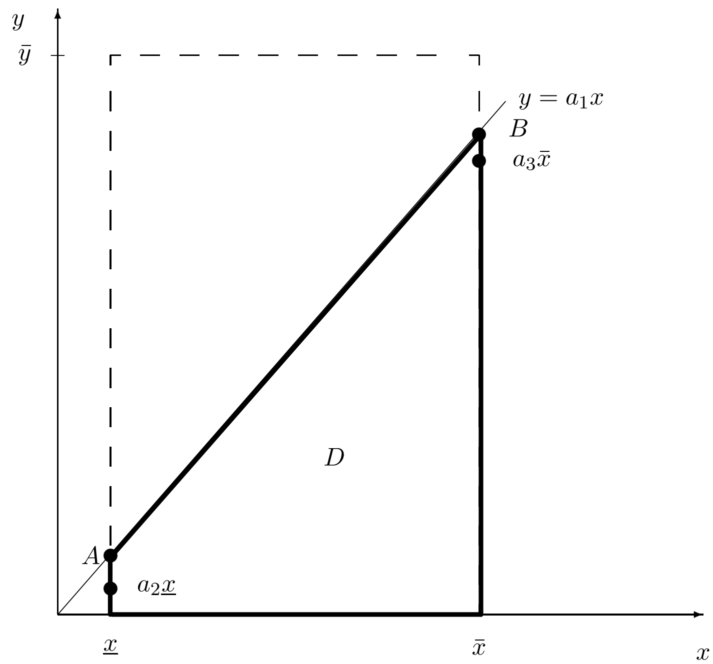
$$D \equiv \Big\{ (x,y) \in M : W_Y(y) - W_X(x) < g^*(x,y) \Big\}.$$

The region D defined in the theorem is called the continuation region in the optimal stopping literature. It divides the state space into two parts. If the current state (x, y) is in D, then the individual's optimal strategy is to continue to stay in the waiting (and thus unemployed) state. The first time that the state crosses the boundary of D is the optimal time for the individual to take the new job and, thus, cease to be in the waiting state. The stopping time τ_D gives exactly the time when the state first crosses the boundary of the continuation region D. Thus, τ_D provides an optimal timing strategy for the individual to terminate the waiting state and start the new job. Intuitively, supposing that currently the state (x, y) is in the continuation region D, if the individual starts the new job, then her utility with the opportunity cost in terms of utility taken into consideration is $W_Y(y) - W_X(x)$; if she behaves optimally and waits, then her expected utility is $\sup_\tau \mathbb{E}^{(x,y)} \{e^{-\rho\tau}[W_Y(Y_\tau) - W_X(X_\tau)]\}$. Since, as claimed in the second part of the theorem, for $(x, y) \in D$, $\sup_\tau \mathbb{E}^{(x,y)} \{e^{-\rho\tau}[W_Y(Y_\tau) - W_X(X_\tau)]\} > W_Y(y) - W_X(x)$, the better choice for her is clearly to wait.

There are two basic cases to be considered in this scenario. In the first case the shape of the continuation region as described by Theorem 1 is given by Figure 1 below.[9]

[9] The points $a_2\underline{x}$ and $a_3\bar{x}$ may lie above the line $y = a_1 x$.

FIG. 1. Continuation Region.

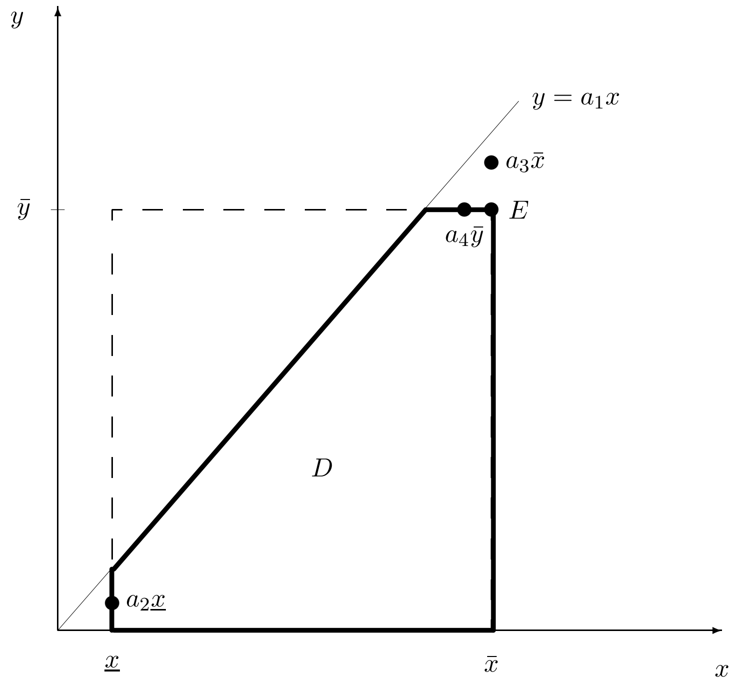


In this case, since $\underline{x}$ and $\bar{x}$ are absorbing boundaries for the stochastic process X_t and since if the process Y_t starts from a state $y > 0$, it will never reach zero, the state (x, y) can never exit from the continuation region D from the lower part of the boundary that lies between point $a_2\underline{x}$ and point $a_3\bar{x}$. If a state (x, y) starts from the interior of D and later hits the vertical line $x = \bar{x}$ below $a_3\bar{x}$, then, as $\bar{x}$ is absorbing, the state will stay in this vertical line and move up and down along the line. Since Y_t can never be zero, eventually the state will exit from the continuation region from point $a_3\bar{x}$. Thus the state can only exit the region D from the upper part of the boundary between the points $a_2\underline{x}$ and $a_3\bar{x}$.

For the second case, if $a_3 < a_1$ and $\bar{y} < a_3\bar{x}$, then the shape of D is given by Figure 2.

There is an absorbing state E in this case. If the state hits the vertical line $x = \bar{x}$, then eventually it will reach the point E. Because $\bar{y}$ is an absorbing boundary of process Y, the state will stay there forever. If the state hits the horizontal line $y = \bar{y}$ between $a_4\bar{y}$ and E, then there is a positive probability that the state will reach state E and stay there forever. Thus, in the case where the new job does not pay high enough income ($\bar{y} < a_3\bar{x}$), there is a positive probability that the individual will

FIG. 2. Continuation Region.



never exit from the waiting state, i.e., $\tau_D = \infty$. In contrast, this will never happen in the first case, where the individual will exit surely at some point.

The boundary of D summarizes the information about the individual's reservation wage for taking the new job. By Theorem 2.1 and the argument above, it is clear that the individual's reservation wage[10] is, in case (i) for example, ax, where $a = a_1$, a_2 and a_3 depending on the values of x. Thus, the reservation wage depends on the level of the unemployment insurance benefits and, in case (ii), on the level of the maximum wage of the new job.

It is worth noticing that it follows from Theorem 1 and the time homogeneity of the diffusion processes X_t and Y_t that the optimal stopping strategy is time homogeneous. This means first that at any time, whether the individual should terminate the waiting state does not depend on how long she has waited before. It only depends on the current state (X_t, Y_t). This should come as no surprise, since an intertemporal additive expected utility is recursive and, therefore, the backward induction method of dynamic programming is directly applicable here. Secondly it means that,

[10] This is the true reservation wage, as opposed to the reservation wage in terms of utility common in the standard search literature.

given a current state (X_t, Y_t), for any s, the probability that the individual will have to wait other s periods is independent of how long she has waited before.[11]

As explained earlier the duration of the waiting state is given by $\min\{\tau^*, \tau_c\}$, where τ_c is the time before the recall. There are two cases that are worthy of special attention here. The first case is where the recall probability is 1. That is, $\tau_c < \infty$ with probability 1. In this case, $\min\{\tau^*, \tau_c\} < \infty$ with probability 1. Thus the individual will always be re-employed, either because of the recall or because she takes a new job. The second case is where the recall probability is less than 1 and the maximum wage from the new job is relatively low (See Figure 2). In this case, if the event that the recall will never occur is independent of the event that the individual never takes a new job, then there is a positive probability that $\min\{\tau^*, \tau_c\} = \infty$. Thus the individual stays in the waiting state forever. This result is consistent with the finding of Pissarides (1982), Katz (1986), Katz and Meyer (1990a,b). The probability of the individual being in the waiting state forever depends, in particular, on the level of the maximum wage from the new job.

3. COMPARATIVE STATICS

Now we turn to the effect of various parameters of the model on the duration of waiting and perform a series of comparative statics on the model.

3.1. The Benchmark Case

First we consider the case where the parameters μ_1 and σ_1 in the stochastic process X_t are identically zero. This is a special case where the unemployment insurance benefits are constant over time, the inflation rate is zero and the individual does not attempt to smooth her consumption over time by saving part of her income from the unemployment insurance (or welfare) benefits. In this case the continuation region D shrinks to a vertical line segment in the (x, y) space with the upper boundary given by

$$a_2 x = \left[\frac{\beta_2(\rho - \mu_2\gamma - \sigma_2^2\gamma(\gamma-1)/2)}{\rho(\beta_2 - \gamma)}\right]^{1/\gamma} x.$$

The effects of the exogenous parameters are summarized in the following proposition:

Proposition 1.

[11] Moreover the probability is independent of the past history of the state over the time interval $[0, t)$.

(a)effect of expected growth rate of income from new job: $da_2/d\mu_2 < 0$.
(b)effect of uncertainty of income from new job: $da_2/d\sigma_2^2 > 0$.
(c)effect of unemployment benefits: $d(a_2x)/dx = a_2 > 0$.
(d)effect of individual's time preference: $da_2/d\rho < 0$.
(e)effect of individual's attitude toward risk: $da_2/d\gamma < 0$.

Clearly the greater a_2 is, the longer is the duration of the waiting state for the individual. So, the results in Proposition 3.1 translate directly into the effect on the duration of waiting of various parameters of the model. For example, part (a) of the proposition states that as the growth rate of the income from the new job increases, the continuation region D becomes smaller. In other words, if the earning prospect from the new job increases, the unemployed individual will terminate the waiting state and start the new job at an earlier date. Parts (d) and (e) of the proposition should be interpreted with caution. The individual's time preference parameter in ρ affects both the utility of the individual derived from the unemployment insurance income and that from the income from the new job. For example a more patient individual derives higher utility from both the unemployment insurance income and the income from the new job. The net effect depends on which part of the utility is affected more. If the increase in the utility from the unemployment insurance income dominates that from the income from the new job, then the more patient individual will wait longer than the less patient individual. Part (d) reports the net effect of ρ in our model. The effect of parameter γ is more complex. Like the parameter ρ, it also affects both the utility from the unemployment insurance income and that from the income from the new job. In addition to that, because γ measures both the relative risk aversion and the rate of intertemporal substitution,[12] a change in γ affects each part of the utility of the individual via the individual's risk aversion parameter and her rate of intertemporal substitution. Part (e) is only the result on the net effect.

3.2. The More General Case

In the more general case in which there is inflation uncertainty, or the unemployed person finds it advantageous to use the financial market, the X_t process is stochastic. In this case the continuation region D takes its full shape. It follows from Theorem 1 that the bigger the a_1, a_2, a_3 and a_4 are, the larger is the region D, and hence the longer the individual will wait, and vice versa. The effect of changes in various parameters on a_2 and a_3 was considered in the previous subsection (Note: $a_2 = a_3$). For the effect on a_1 and a_4 we have the following two propositions.

[12]See Epstein and Zin (1989) and the reference therein.

Proposition 2.

(a)effect of expected growth rate of income generated from UI: $da_1/d\mu_1 > 0$.

(b)effect of expected growth rate of income from new job: $da_1/d\mu_2 < 0$.

(c)effect of uncertainty of income generated from UI: If $\mu_2 - \mu_1 \geq \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}(3-2\gamma)$, *then* $da_1/d\sigma_1^2 > 0$. *If* $\mu_2 - \mu_1 < \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}(3-2\gamma)$, *then* $da_1/d\sigma_1^2 > 0$ *if and only if* $\rho > \mu_2(\gamma-1) + \frac{\sigma_2^2}{2}(1-\gamma)(2-\gamma) + \mu_1$.

(d)effect of uncertainty of income from new job: $da_1/d\sigma_2^2 > 0$.

(e)effect of individual's time preference: $da_1/d\rho < 0$.

Proposition 3.

(a)effect of expected growth rate of income generated from UI: $da_4/d\mu_1 > 0$.

(b)effect of uncertainty of income generated from UI: $da_4/d\sigma_1^2 < 0$.

(c)effect of individual's time preference: $da_4/d\rho < 0$.

(d)effect of individual's attitude toward risk: $da_4/d\gamma < 0$.

The effect of γ is, in general, indeterminate. As explained earlier, a change in γ affects both parts of the individual's utility, the part before employment and the part after employment, and it affects each part through the risk aversion parameter and the rate of intertemporal substitution. As in the bench-mark case where the unemployment insurance income is deterministic, we are able to sign the effect of γ on a_2 and a_3. However, because of the extra uncertainty on the income from unemployment insurance (or welfare), the net effect of γ on a_1 becomes indeterminate. Part (d) of Proposition 3 suggests that the net effect of γ on a_4 is just the opposite of that on a_2. The result that the larger the unemployment insurance payment is, the longer is the duration of wait (Proposition 1 (c), Proposition 2. (a) and Proposition 3 (a)) is consistent with the empirical finding reported by Moffit and Nicholson (1982), and Mortensen (1977).

4. AN EXTENSION OF THE BASIC MODEL

In this section we extend the model in section 2 to a more general setting. Specifically the individual's utility function $u(x)$ is assumed only to be continuous for $x > 0$. The diffusion processes X_t and Y_t are assumed to evolve according to

$$dX_t = \mu_1(X_t)X_t dt + \sigma_{11}(X_t)X_t dB_t^1 + \cdots + \sigma_{1n}(X_t)X_t dB_t^n, \quad X_0 = x > 0, \tag{16}$$

$$dY_t = \mu_2(Y_t)Y_t dt + \sigma_{21}(Y_t)Y_t dB_t^1 + \cdots + \sigma_{2n}(Y_t)Y_t dB_t^n, \quad Y_0 = y > 0, \tag{17}$$

where μ_1 and μ_2 are the growth rate parameters of the processes X_t and Y_t, $\sigma_{11}, \ldots, \sigma_{2n}$ are the variance parameters, and $B^1, \ldots, B^n$ are n standard Brownian motions, which may be correlated, with $\eta = (\eta_{ij})$ being the correlation coefficient matrix for the n–dimensional Brownian motion. It is assumed that μ_i and σ_{ij} are bounded continuous functions. It is also assumed, as usual, that $\mu_i(x)x$ and $\sigma_{ij}(x)x$ are Lipschitz continuous.

Again we consider the processes with values in some bounded intervals, $X_{t\wedge\tau_{(0,\bar{x})}}$ and $Y_{t\wedge\tau_{(0,\bar{y})}}$, and denote them by X_t and Y_t with some abuse of notation. The generators of these two processes are given by

$$\begin{aligned}
\mathcal{A}_X &= \mu_1(x)x\frac{\partial}{\partial x} + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{1i}(x)\eta_{ij}\sigma_{1j}(x)x^2\frac{\partial^2}{\partial x^2}, && (18)\\
&\quad \text{if } 0 < x < \bar{x}, \text{ otherwise } \mathcal{A}_X = 0 \\
\mathcal{A}_Y &= \mu_2(y)y\frac{\partial}{\partial y} + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{2i}(y)\eta_{ij}\sigma_{2j}(y)y^2\frac{\partial^2}{\partial y^2}. && (19)\\
&\quad \text{if } 0 < y < \bar{y}, \text{ otherwise } \mathcal{A}_Y = 0
\end{aligned}$$

Define the functions W_X and W_Y in an analogous way as in section 2 with the differential operators $\mathcal{A}_X$ and $\mathcal{A}_Y$ replaced by the ones defined above.

THEOREM 2. *Let*

$$g^*(x,y) = \sup_{\tau} \mathbb{E}^{(x,y)}\left\{e^{-\rho\tau}[W_Y(Y_\tau) - W_X(X_\tau)]\right\}, \tag{20}$$

where the supremum is taken over all $\{\mathcal{F}_t\}$*–stopping times. Let* $M = (0,\bar{x}] \times (0,\bar{y}]$*. Define*[13]

$$D \equiv \Big\{(x,y) \in M : W_Y(y) - W_X(x) < g^*(x,y)\Big\} \qquad (\textit{continuation region }).$$

Then (i) D has a nonempty interior; (ii)

$$\tau_D = \inf\{t > 0 : (X_t, Y_t) \notin D\}.$$

[13] It is shown in the proof of Theorem 1 that this definition is consistent with that in Theorem 1.

is an optimal stopping time for the individual's problem, i.e., it solves (20); and (iii) if τ^* *is another optimal stopping time for the optimal stopping problem above, then* $\tau^* \geq \tau_D$.

The first claim of the theorem establishes the existence of the waiting phenomenon, i.e., for some states the optimal waiting time τ^* is strictly positive and the set of these states has a positive Lebesgue measure. The second claim gives an analytic expression for an optimal stopping strategy, τ_D. Finally the third claims says that the optimal stopping strategy τ_D gives the shortest waiting time among all of the optimal stopping strategies and, thus, gives the lower bound for all optimal waiting times.

5. CONCLUSIONS

A simplifying assumption made in the paper is that if the unemployed person does not take the job immediately, the job opportunity will not vanish. That is, the employer is willing to keep the job open to the individual forever. This is clearly a very strong assumption from a practical point of view. This assumption, however, is made only to focus our attention on the existence of the waiting phenomenon. For our model it suffices to assume that the individual is interested in a class of jobs that is always available to her. If the individual does not take a particular job, it may be taken by someone else. This is, however, immaterial to the individual. As long as the class of preferred jobs is available, the individual can choose to take or not to take a particular job that she is facing at a point in time. Interpreted in this way, the process Y represents the generic income from this class of jobs.

It is hoped that the theoretical analysis provided in this paper adds to our understanding on how risk aversion on the part of an unemployed individual and how uncertainty faced by this individual affect the duration of wait unemployment on her part. This, in turn, is hoped to add to the list of information useful for policy makers who are interested in further improving the unemployment insurance scheme.

Finally, although throughout the paper we focus on the issue of the wait unemployment, the model built in this paper is also applicable to the study of job switching.[14] If we look at the two phenomena from the perspective of switching from one income stream to another, there is no fundamental theoretical difference between the job-switching case, where an individual who currently holds a job considers whether and when to switch from the current job to a potentially better paid new job (note that the current job and the new job can be located in the urban area, or the current job is

[14] For applications of job switching models, see Dixit and Rob (1994a,b).

located in the rural area and the new job is located in the urban area), and the wait phenomenon, where an unemployed person gives up unemployment insurance (or welfare) benefits to receive an income stream from the new job.

APPENDIX A

To avoid unnecessary repetitions in the proof steps, we provide proofs for Section 4 first collected in Appendix A, followed by proofs for propositions in Section 2 contained in Appendix B, and then proofs for proposition in Section 3 presented in Appendix C.

PROOFS FOR SECTION 4

Note that some of the claims made around equations (12) and (13) are special cases of the lemma here. The following lemma is stated for the function W_X. A similar result holds true for the function W_Y. Note that $\underline{x}$ may be zero.

Lemma 1. *Let W be the solution to the following problem:*

$$
\begin{aligned}
&-\rho W_X + \mu(x)x\frac{\partial W_X}{\partial x} + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_{1i}(x)\eta_{ij}\sigma_{1j}(x)x^2\frac{\partial^2 W_X}{\partial x^2} + u(x) = 0 \\
&\quad \underline{x} < x < \bar{x} \\
&\lim_{x\to\underline{x}} W_X(x) = \int_0^\infty e^{-\rho t}u(\underline{x})\ dt \\
&\lim_{x\to\bar{x}} W_X(x) = \int_0^\infty e^{-\rho t}u(\bar{x})\ dt.
\end{aligned}
$$

Then for $x \in [\underline{x}, \bar{x}]$, we have

$$
W_X(x) = \mathbb{E}^x\left\{\int_0^\infty e^{-\rho t}u(X_t')\ dt\right\}.
$$

Proof. Let $\tau = \inf\{t > 0 : X_t \notin (\underline{x}, \bar{x})\}$. Then by Ito's formula,

$$
\begin{aligned}
&e^{-\rho(t\wedge\tau)}W_X(X_{t\wedge\tau}) - W_X(x) \\
= &\int_0^{t\wedge\tau} e^{-\rho s}\left(-\rho W_X(X_s) + \mathcal{A}_X W_X(X_s)\right)\ ds \\
+ &\int_0^{t\wedge\tau} e^{-\rho s}\sum_{j=1}^{n}\sigma_{1j}(X_s)X_s\frac{\partial W_X}{\partial x}(X_s)\ dB_s.
\end{aligned}
$$

Taking expectation on both sides of the above equation and noting that the second term on the right hand side of the equation is a martingale, we have

$$\begin{aligned} & \mathbb{E}^x \left\{ e^{-\rho(t\wedge\tau)} W_X(X_{t\wedge\tau}) \right\} - W_X(x) \\ = \ & \mathbb{E}^x \left\{ \int_0^{t\wedge\tau} e^{-\rho s} \left(-\rho W_X(X_s) + \mathcal{A}_X W_X(X_s) \right) \, ds \right\}. \end{aligned}$$

Since W_X is the solution to the differential equation, we have

$$\begin{aligned} & \mathbb{E}^x \left\{ e^{-\rho(t\wedge\tau)} W_X(X_{t\wedge\tau}) \right\} - W_X(x) \\ = \ & \mathbb{E}^x \left\{ \int_0^{t\wedge\tau} -e^{-\rho s} u(X_s) \, ds \right\}. \end{aligned}$$

Finally, letting $t \to \infty$, we have

$$\begin{aligned} W_X(x) \ &= \ \mathbb{E}^x \left\{ e^{-\rho\tau} W_X(X_\tau) \right\} + \mathbb{E}^x \left\{ \int_0^\tau e^{-\rho s} u(X_s) \, ds \right\} \\ &= \ \mathbb{E}^x \left\{ \int_0^\infty e^{-\rho s} u(X_s') \, ds \right\}. \end{aligned}$$

▮

Proof of Theorem 2: The second part of the theorem is by Theorem 10.9 of Oksendal (1992). Note that since the function $e^{-\rho t}[W_Y(y) - W_X(x)]$ is bounded, the conditions in that theorem that $e^{-\rho t}[W_Y(y) - W_X(x)]$ is nonnegative and that $\tau_D < \infty$ a.s. can be removed. The third part of the theorem is by Theorem 10.12 of Oksendal (1992).

For the first part, by Oksendal (1992, p.166), the continuation region D includes the set

$$\begin{aligned} & \{(x, y) : -\rho W_Y(y) + \mathcal{A}_Y W_Y(y) - (-\rho W_X(x) + \mathcal{A}_X W_X(x)) > 0\} \\ = \ & \{(x, y) : u(x) - u(y) > 0\} = \{(x, y) : x > y\} \end{aligned}$$

which is open and nonempty. ▮

APPENDIX B

PROOFS FOR SECTION 2

The next lemma provides the close form expression for W_X and W_Y in section 2.

LEMMA 2. *The differential equation*

$$\begin{aligned} -\rho W + \mu z \frac{\partial W}{\partial z} + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 W}{\partial z^2} + u(z) &= 0 \quad \underline{z} < z < \bar{z} \\ \lim_{z \to \underline{z}} W(z) &= \int_0^\infty e^{-\rho t} u(\underline{z})\, dt \\ \lim_{z \to \bar{z}} W(z) &= \int_0^\infty e^{-\rho t} u(\bar{z})\, dt \end{aligned}$$

has the unique solution

$$W(z) = C_1 z^{\alpha_1} + C_2 z^{\alpha_2} + C_3 z^{\gamma},$$

where

$$\begin{aligned} \alpha_1 &= \frac{1}{\sigma^2}\left(-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}\right) \\ \alpha_2 &= \frac{1}{\sigma^2}\left(-\left(\mu - \frac{1}{2}\sigma^2\right) - \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}\right) \\ C_3 &= \left[\gamma\left(\rho - \mu\gamma - \frac{1}{2}\sigma^2\gamma(\gamma-1)\right)\right]^{-1} \\ C_1 &= \begin{cases} (1/\rho - C_3)\left(\underline{z}^{\gamma}\bar{z}^{\alpha_2} - \bar{z}^{\gamma}\underline{z}^{\alpha_2}\right)/\left(\underline{z}^{\alpha_1}\bar{z}^{\alpha_2} - \bar{z}^{\alpha_1}\underline{z}^{\alpha_2}\right) & \text{if } \underline{z} \neq 0 \\ [1/\rho - C_3]\, \bar{z}^{\gamma - \alpha_1} & \text{if } \underline{z} = 0 \end{cases} \\ C_2 &= \begin{cases} (1/\rho - C_3)\left(\bar{z}^{\gamma}\underline{z}^{\alpha_1} - \underline{z}^{\gamma}\bar{z}^{\alpha_1}\right)/\left(\underline{z}^{\alpha_1}\bar{z}^{\alpha_2} - \bar{z}^{\alpha_1}\underline{z}^{\alpha_2}\right) & \text{if } \underline{z} \neq 0 \\ 0 & \text{if } \underline{z} = 0. \end{cases} \end{aligned}$$

Furthermore, for $z \in [\underline{z}, \bar{z}]$,

$$W(z) = \mathbb{E}^z\left\{\int_0^\infty e^{-\rho t} u(Z'_t)\, dt\right\},$$

where Z' *can be either* X' *or* Y'.

Proof. The second part of the lemma follows from Lemma A.1. The first part can be readily verified. ∎

COROLLARY 1. *:* $C_1 z^{\alpha_1} + C_2 z^{\alpha_2}$ *is a solution to the homogeneous differential equation*

$$-\rho W + \mu z \frac{\partial W}{\partial z} + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 W}{\partial z^2} = 0, \qquad z \in (\underline{z}, \bar{z}).$$

Lemma 3. *The partial differential equation*

$$\begin{aligned}
&\frac{\partial g}{\partial t} + \mu_1 x \frac{\partial g}{\partial x} + \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 g}{\partial x^2} + \mu_2 y \frac{\partial g}{\partial y} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 g}{\partial y^2} = 0 \\
&\underline{x} < x < \bar{x},\ 0 < y < \bar{y} \\
&\lim_{y \to 0} g(0, x, y) = \text{bounded}
\end{aligned}$$

has a family of solutions

$$g(t, x, y) = e^{-\rho t}\left[C x^{\alpha} y^{\beta}\right],$$

where α and β satisfy the following elliptic relation,

$$\begin{aligned}
& \frac{\sigma_1^2}{2}\left(\alpha - \left(\frac{1}{2} - \frac{\mu_1}{\sigma_1^2}\right)\right)^2 + \frac{\sigma_2^2}{2}\left(\beta - \left(\frac{1}{2} - \frac{\mu_2}{\sigma_2^2}\right)\right)^2 \\
= \ & \frac{\sigma_1^2}{2}\left(\frac{1}{2} - \frac{\mu_1}{\sigma_1^2}\right)^2 + \frac{\sigma_2^2}{2}\left(\frac{1}{2} - \frac{\mu_2}{\sigma_2^2}\right)^2 + \rho,
\end{aligned}$$

with β being positive.

Proof. By setting $g(t, x, y) = e^{-\rho t}\phi(x, y)$, $f(u, v) = \phi(e^u, e^v)$, $x = e^u$ and $y = e^v$, we can transform the original partial differential equation to

$$-\rho f + \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\frac{\partial f}{\partial u} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 f}{\partial u^2} + \left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\frac{\partial f}{\partial v} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 f}{\partial v^2} = 0.$$

Put $f(u, v) = e^{\alpha u + \beta v}$. Then we have

$$-\rho + \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\alpha + \frac{1}{2}\sigma_1^2\alpha^2 + \left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\beta + \frac{1}{2}\sigma_2^2\beta^2 = 0,$$

which leads to the elliptic relation

$$\begin{aligned}
& \frac{\sigma_1^2}{2}\left(\alpha - \left(\frac{1}{2} - \frac{\mu_1}{\sigma_1^2}\right)\right)^2 + \frac{\sigma_2^2}{2}\left(\beta - \left(\frac{1}{2} - \frac{\mu_2}{\sigma_2^2}\right)\right)^2 \\
= \ & \frac{\sigma_1^2}{2}\left(\frac{1}{2} - \frac{\mu_1}{\sigma_1^2}\right)^2 + \frac{\sigma_2^2}{2}\left(\frac{1}{2} - \frac{\mu_2}{\sigma_2^2}\right)^2 + \rho.
\end{aligned}$$

The boundary condition rules out the negative β. ∎

Proof of Theorem 1: The proof consists of three steps. The first step is the construction of a candidate function $e^{-\rho t}H(x,y)$ as the least superharmonic majorant of the function $e^{-\rho t}(W_Y(y) - W_X(x))$.[1] The second step shows that the function $e^{-\rho t}H(x,y)$ is superharmonic. The third step shows that it is a majorant of $e^{-\rho t}(W_Y(y) - W_X(x))$. It then follows from Oksendal (1992, Chapter 10) that $H(x,y) = g^*(x,y)$, i.e.,

$$H(x,y) = \sup_{\tau} \mathbb{E}^{(x,y)}\{e^{-\rho\tau}[W_Y(Y_\tau) - W_X(X_\tau)]\},$$

and τ_D is the solution to (14). The second part of the theorem is a corollary of the third step of the proof.
Step 1: Consider first the following problem: find a pair $(b,h) \in \mathbb{R} \times C^2((\underline{x},\bar{x}) \times (0,\bar{y}))$ such that

$$\begin{aligned}
-\rho h + \mathcal{A}_X h + \mathcal{A}_Y h &= 0 && \text{(B.1)}\\
\lim_{y\to 0} h(x,y) &= \text{bounded} && \text{(B.2)}\\
\frac{\partial h}{\partial x} &= -\frac{\partial W_X}{\partial x} \quad \text{for } y = bx && \text{(B.3)}\\
\frac{\partial h}{\partial y} &= \frac{\partial W_Y}{\partial y} \quad \text{for } y = bx && \text{(B.4)}\\
h(x,y) &= W_Y(y) - W_X(x) \quad \text{for } y = bx. && \text{(B.5)}
\end{aligned}$$

Using Lemma B.3, it can be verified that the pair of $b = a_1$, where a_1 is as specified in the statement of the theorem, and the function

$$\begin{aligned}
h_1(x,y) &= B_1 x^{\gamma-\beta_1} y^{\beta_1} + W_Y(y) - \frac{y^\gamma}{\gamma[\rho - \mu_2\gamma - \sigma_2^2\gamma(\gamma-1)/2]}\\
&\quad -W_X(x) + \frac{x^\gamma}{\gamma[\rho - \mu_1\gamma - \sigma_1^2\gamma(\gamma-1)/2]},
\end{aligned}$$

where β_1 is as specified in the statement of the theorem,

$$B_1 = \frac{1}{(\beta_1 - \gamma)a_1^{\beta_1}(\rho - \mu_1\gamma - \sigma_1^2\gamma(\gamma-1)/2)},$$

satisfies the partial differential equation and conditions (23) and (24). Since

$$\frac{\partial h_1}{\partial x}x + \frac{\partial h_1}{\partial y}y = \gamma(h(x,y) - W_Y(y) + W_X(x)) - \frac{\partial W_X}{\partial x}x + \frac{\partial W_Y}{\partial y}y,$$

[1] See Oksendal (1992) for definitions of superharmonic functions and superharmonic majorants of Borel functions. See also Dynkin (1965, II).

the last boundary condition is also satisfied because of the two conditions in (23) and (24). Therefore (a_1, h_1) solves the problem. Note that the expression for β_1 is obtained by substituting the relation $\alpha + \beta_1 = \gamma$ into the elliptic relation

$$\begin{aligned} & \frac{\sigma_1^2}{2}\left(\alpha - \left(\frac{1}{2} - \frac{\mu_1}{\sigma_1^2}\right)\right)^2 + \frac{\sigma_2^2}{2}\left(\beta_1 - \left(\frac{1}{2} - \frac{\mu_2}{\sigma_2^2}\right)\right)^2 \\ = & \frac{\sigma_1^2}{2}\left(\frac{1}{2} - \frac{\mu_1}{\sigma_1^2}\right)^2 + \frac{\sigma_2^2}{2}\left(\frac{1}{2} - \frac{\mu_2}{\sigma_2^2}\right)^2 + \rho, \end{aligned}$$

which leads to

$$\frac{\sigma_1^2 + \sigma_2^2}{2}\beta_1^2 - \left[\sigma_1^2\gamma - \left(\frac{\sigma_1^2}{2} - \mu_1\right) + \left(\frac{\sigma_2^2}{2} - \mu_2\right)\right]\beta_1 - \left[\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma - 1)\right] = 0.$$

The positivity of β_1 then leads to

$$\begin{aligned} \beta_1 = & \frac{\sqrt{\left[\sigma_1^2\gamma - \left(\frac{\sigma_1^2}{2} - \mu_1\right) + \left(\frac{\sigma_2^2}{2} - \mu_2\right)\right]^2 + 2(\sigma_1^2 + \sigma_2^2)\left[\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma - 1)\right]}}{\sigma_1^2 + \sigma_2^2} \\ & + \frac{\sigma_1^2\gamma - \left(\frac{\sigma_1^2}{2} - \mu_1\right) + \left(\frac{\sigma_2^2}{2} - \mu_2\right)}{\sigma_1^2 + \sigma_2^2}. \end{aligned}$$

Secondly consider the following problem: find a pair $(b, h) \in \mathbb{R} \times C^2((0, \bar{y}))$ such that

$$\begin{aligned} -\rho h + \mathcal{A}_Y h &= 0 && \text{(B.6)} \\ \lim_{y \to 0} h(y) &= \text{bounded} && \text{(B.7)} \\ \frac{dh}{dy} &= \frac{dW_Y(y)}{dy} \quad \text{at } y = b\underline{x} && \text{(B.8)} \\ h(y) &= W_Y(y) - W_X(\underline{x}) \quad \text{at } y = b\underline{x}. && \text{(B.9)} \end{aligned}$$

Let $b = a_2$ be as specified in the theorem and

$$h_2(y) = B_2 y^{\beta_2},$$

where β_2 is as specified in the theorem and

$$B_2 = \frac{W_Y(a_2\underline{x}) - W_X(\underline{x})}{(a_2\underline{x})^{\beta_2}}.$$

Using Lemma B.1 and Corollary B.2, it can be verified that (a_2, h_2) solves the problem.

Thirdly consider the following problem: find a pair $(b, h) \in \mathbb{R} \times C^2((0, \bar{y}))$ such that

$$\begin{aligned} -\rho h + \mathcal{A}_Y h &= 0 && \text{(B.10)} \\ \lim_{y \to 0} h(y) &= \text{bounded} && \text{(B.11)} \\ \frac{dh}{dy} &= \frac{dW_Y(y)}{dy} \quad \text{at } y = b\bar{x} && \text{(B.12)} \\ h(y) &= W_Y(y) - W_X(\bar{x}) \quad \text{at } y = b\bar{x}. && \text{(B.13)} \end{aligned}$$

Let $b = a_3$ and

$$h_3(y) = B_3 y^{\beta_3},$$

where β_3 is as specified in the theorem and

$$B_3 = \frac{W_Y(a_3\bar{x}) - W_X(\bar{x})}{(a_3\bar{x})^{\beta_3}}.$$

It can be verified that (a_3, h_3) solves the problem.

Finally consider the following problem: find a pair $(b, h) \in \mathbb{R} \times C^2([\underline{x}, \bar{x}])$ such that

$$\begin{aligned} -\rho h + \mathcal{A}_X h &= 0 && \text{(B.14)} \\ \frac{dh}{dx} &= -\frac{dW_X(x)}{dx} \quad \text{at } x = b\bar{y} && \text{(B.15)} \\ h(x) &= W_Y(\bar{y}) - W_X(x) \quad \text{at } x = b\bar{y}. && \text{(B.16)} \end{aligned}$$

Let $b = a_4$ and

$$h_4(x) = B_4 x^{\beta_4} - C_1 x^{\alpha_1},$$

where β_4 is as specified in the theorem,

$$B_4 = \frac{W_Y(\bar{y}) - W_X(a_4\bar{y}) + C_1(a_4\bar{y})^{\alpha_1}}{(a_4\bar{y})^{\beta_4}},$$

and C_1 and α_1 are as defined in Lemma B.1 with μ and σ replaced by μ_1 and σ_1. It can be verified that (a_4, h_4) solves the problem.

Case 1: $\bar{y} > \max\{a_1\bar{x}, a_3\bar{x}\}$. In this case the continuation region will lie in the set $[\underline{x}, \bar{x}] \times (0, \bar{y})$, as will be shown below. So we will consider the states (x, y) such that $y < \bar{y}$. Let $M_1 = \{(x, y) \in M : x \in (\underline{x}, \bar{x}), y \in (0, \bar{y}),$ and $y < a_1 x\}$, $M_2 = \{(x, y) \in M : x \in (\underline{x}, \bar{x}), y \in (0, \bar{y}),$ and $y \geq a_1 x\}$, $M_3 = \{(x, y) \in M : x = \underline{x}, y \in (0, \bar{y}),$ and $y < a_2\underline{x}\}$, $M_4 = \{(x, y) \in$

$M : x = \underline{x}, y \in (0, \bar{y}),$ and $y \geq a_2\underline{x}\}$, $M_5 = \{(x, y) \in M : x = \bar{x}, y \in (0, \bar{y}),$ and $y < a_3\bar{x}\}$, $M_6 = \{(x, y) \in M : x = \bar{x}, y \in (0, \bar{y}),$ and $y \geq a_3\bar{x}\}$. Define a function H on $M_1 \cup \cdots \cup M_6$ by

$$H(x,y) = \begin{cases} h_1(x,y) & \text{if } (x,y) \in M_1, \\ W_Y(y) - W_X(x) & \text{if } (x,y) \in M_2, \\ h_2(x,y) & \text{if } (x,y) \in M_3, \\ W_Y(y) - W_X(\underline{x}) & \text{if } (x,y) \in M_4, \\ h_3(x,y) & \text{if } (x,y) \in M_5, \\ W_Y(y) - W_X(\bar{x}) & \text{if } (x,y) \in M_6. \end{cases}$$

Case 2: $\bar{y} \leq \max\{a_1\bar{x}, a_3\bar{x}\}$. Define $M_7 = \{(x, y) \in M : y = \bar{y}, x \in [\underline{x}, \bar{x}],$ and $x < a_4\bar{y}\}$, $M_8 = \{(x, y) \in M : y = \bar{y}, x \in [\underline{x}, \bar{x}],$ and $x \geq a_4\bar{y}\}$. Note that $M = M_1 \cup \cdots \cup M_8$. Define a function H on $M_1 \cup \cdots \cup M_8$ by

$$H(x,y) = \begin{cases} h_1(x,y) & \text{if } (x,y) \in M_1, \\ W_Y(y) - W_X(x) & \text{if } (x,y) \in M_2, \\ h_2(x,y) & \text{if } (x,y) \in M_3, \\ W_Y(y) - W_X(\underline{x}) & \text{if } (x,y) \in M_4, \\ h_3(x,y) & \text{if } (x,y) \in M_5, \\ W_Y(y) - W_X(\bar{x}) & \text{if } (x,y) \in M_6 \\ W_Y(\bar{y}) - W_X(x) & \text{if } (x,y) \in M_7 \\ h_4(x,y) & \text{if } (x,y) \in M_8. \end{cases}$$

Step 2: We show that the function $e^{-\rho t}H(x, y)$ is superharmonic, i.e., for any stopping time τ,

$$e^{-\rho t}H(x,y) \geq \mathbb{E}^{(t,x,y)}\left(e^{-\rho(t+\tau)}H(X_\tau, Y_\tau)\right).$$

Since $e^{-\rho t}H(x, y)$ is bounded from below, by Dynkin (1965, II, pp. 4, 15, 16, 22), it is sufficient to show that for each point (t, x, y), there exists an open set U such that for each open subset V of U, with $(t, x, y) \in V$,

$$e^{-\rho t}H(x,y) \geq \mathbb{E}^{(x,y)}\left(e^{-\rho(t+\tau_V)}H(X_{\tau_V}, Y_{\tau_V})\right), \tag{B.17}$$

where τ_V is the first exit time from V. Note that the generator $\mathcal{A}$ of the process (t, X_t, Y_t) is given by, for any suitably differentiable function $f(t, x, y)$,

$$\mathcal{A}f = \begin{cases} \frac{\partial}{\partial t}f + \mathcal{A}_X f + \mathcal{A}_Y f & \text{if } \underline{x} < x < \bar{x}, 0 < y < \bar{y}; \\ \frac{\partial}{\partial t}f + \mathcal{A}_Y f & \text{if } x = \underline{x}, 0 < y < \bar{y}; \\ \frac{\partial}{\partial t}f + \mathcal{A}_Y f & \text{if } x = \bar{x}, 0 < y < \bar{y}; \\ \frac{\partial}{\partial t}f + \mathcal{A}_X f & \text{if } y = \bar{y}, \underline{x} < x < \bar{x}. \end{cases}$$

Case 1: If $\bar{y} > \max\{a_1\bar{x}, a_3\bar{x}\}$, then we do not have to be concerned with the case of $y = \bar{y}$. Let F^o denote the interior of the set F. Applying the generator to $e^{-\rho t}H(x,y)$ on $\mathbb{R} \times M_2^o$, we have

$$\begin{aligned}
&\mathcal{A}(e^{-\rho t}H(x,y)) \\
= \ & -\rho\left[W_Y(y) - W_X(x)\right]e^{-\rho t} + \mathcal{A}_X\left[W_Y(y) - W_X(x)\right]e^{-\rho t} \\
& + \mathcal{A}_Y\left[W_Y(y) - W_X(x)\right]e^{-\rho t} \\
= \ & \left[-\rho W_Y(y) + \mathcal{A}_Y W_Y(y) + \rho W_X(x) - \mathcal{A}_X W_X(x)\right]e^{-\rho t} \\
= \ & (-u(y) + u(x))e^{-\rho t}
\end{aligned}$$

By expression (42) in Appendix C, $a_1 > 1$. Thus $\mathcal{A}e^{-\rho t}H(x,y) < 0$ on $\mathbb{R} \times M_2^o$. By construction, $\mathcal{A}e^{-\rho t}H(x,y) = 0$ on $\mathbb{R} \times M_1$. Therefore $\mathcal{A}e^{-\rho t}H(x,y) \le 0$ on $\mathbb{R} \times (M_1 \cup M_2^o)$. Now by Dynkin's formula,

$$\begin{aligned}
\mathbb{E}^{(x,y)}\left[e^{-\rho(t+\tau)}H(X_\tau, Y_\tau)\right] &= e^{-\rho t}H(x,y) + \mathbb{E}^{(x,y)}\left[\int_0^{\tau_V} \mathcal{A}(e^{-\rho t}H(x,y))\, dt\right] \\
&\le e^{-\rho t}H(x,y),
\end{aligned}$$

for any $V \subset \mathbb{R} \times (M_1 \cup M_2^o)$, which implies that (37) holds.

For (x,y) such that $y = a_1 x$, by a generalized Dynkin's formula (Brekke and Oksendal (1991)), (37) again holds true.

On $\mathbb{R} \times M_4^o$,

$$\begin{aligned}
\mathcal{A}(e^{-\rho t}H(x,y)) &= -\rho[W_Y(y) - W_X(\underline{x})] + \mathcal{A}_Y W_Y(y) \\
&= -u(y) + \rho W_X(\underline{x}) \le \frac{-1}{\gamma}\underline{x}^\gamma a_2^\gamma + \frac{\underline{x}^\gamma}{\gamma} < 0,
\end{aligned}$$

because by expression (43) in Appendix C, $a_2 > 1$. Hence $\mathcal{A}e^{-\rho t}H(x,y) < 0$ on $\mathbb{R} \times M_4^o$. Again by construction, $\mathcal{A}e^{-\rho t}H(x,y) = 0$ on $\mathbb{R} \times M_3$. Therefore $\mathcal{A}e^{-\rho t}H(x,y) \le 0$ on $\mathbb{R} \times (M_3 \cup M_4^o)$. By Dynkin's formula, (37) holds true for $(t,x,y) \in \mathbb{R} \times (M_3 \cup M_4^o)$. For point $(t, \underline{x}, a_2\underline{x})$, the same generalized Dynkin's formula implies that (37) holds true.

By the same argument, (37) holds true for $(t,x,y) \in \mathbb{R} \times (M_5 \cup M_6)$. Therefore $e^{-\rho t}H(x,y)$ is superharmonic on $\mathbb{R} \times M$.

Case 2: $\bar{y} > \max\{a_1\bar{x}, a_3\bar{x}\}$. It is clear that we only need to show that $\mathcal{A}(e^{-\rho t}H(x,y)) \le 0$ on $\mathbb{R} \times (M_7 \cup M_8)$ and $H(x,y) > W_Y(y) - W_X(x)$ on M_8. On M_7^o,

$$\mathcal{A}(e^{-\rho t}H(x,y)) = -\rho[W_Y(\bar{y}) - W_X(x)] - \mathcal{A}_X W_X(x) = -u(\bar{y}) + u(x) < 0,$$

because by expression (44)in Appendix C, $a_4 < 1$. Hence $\mathcal{A}e^{-\rho t}H(x,y) < 0$ on $\mathbb{R} \times M_7^o$. By construction, $\mathcal{A}e^{-\rho t}H(x,y) = 0$ on $\mathbb{R} \times M_8$. Therefore

$\mathcal{A}e^{-\rho t}H(x,y) \leq 0$ on $\mathbb{R} \times (M_8 \cup M_7^o)$. By Dynkin's formula, (37) holds true for $(x,y) \in \mathbb{R} \times (M_8 \cup M_7^o)$. For point $(t, a_4\bar{y}, \bar{y})$, the generalized Dynkin's formula implies that (37) holds true. Therefore $e^{-\rho t}H(x,y)$ is superharmonic on $\mathbb{R} \times M$.

Step 3: We show that $e^{-\rho t}H(x,y)$ is a superharmonic majorant of $e^{-\rho t}(W_Y(y) - W_X(x))$, i.e.,

$$e^{-\rho t}H(x,y) \geq e^{-\rho t}(W_Y(y) - W_X(x)),$$

or equivalently,

$$H(x,y) \geq W_Y(y) - W_X(x).$$

Case 1: This inequality is clearly true on $M_2 \cup M_4 \cup M_6$. On M_1,

$$\begin{aligned}
& H(x,y) - (W_Y(y) - W_X(x)) \\
= \;& B_1 x^{\gamma-\beta_1} y^{\beta_1} - \frac{y^\gamma}{\gamma[\rho - \mu_2\gamma - \sigma_2^2\gamma(\gamma-1)/2]} + \frac{x^\gamma}{\gamma[\rho - \mu_1\gamma - \sigma_1^2\gamma(\gamma-1)/2]} \\
= \;& \left[B_1 b^{\beta_1} - \frac{b^\gamma}{\gamma[\rho - \mu_2\gamma - \sigma_2^2\gamma(\gamma-1)/2]} + \frac{1}{\gamma[\rho - \mu_1\gamma - \sigma_1^2\gamma(\gamma-1)/2]}\right] x^\gamma,
\end{aligned}$$

where $b = y/x$. Consider the term inside the bracket of the above expression. It is zero at $b = a$ by condition stated in (21)-(25) in Step 1. Its derivative with respect to b is

$$B_1\beta_1 b^{\beta_1 - 1} - \frac{b^{\gamma-1}}{[\rho - \mu_2\gamma - \sigma_2^2\gamma(\gamma-1)/2]}.$$

It is at zero if and only if

$$b^{\gamma-\beta_1} = B_1\beta_1[\rho - \mu_2\gamma - \sigma_2^2\gamma(\gamma-1)/2] = a_1^{\gamma-\beta_1},$$

i.e., $b = a_1$. Since $\lim_{y\to 0} H(x,y) - (W_Y(y) - W_X(x)) = W_X(x) > 0$, we must have

$$H(x,y) - (W_Y(y) - W_X(x)) > 0 \qquad \text{on } M_1. \tag{B.18}$$

On M_3, we have

$$\begin{aligned}
H(x,y) - [W_Y(y) - W_X(x)] &= h_2(y) - (W_Y(y) - W_X(\underline{x})) \\
&= B_2 y^{\beta_2} - C_1 y^{\beta_2} - C_3 y^\gamma + W_X(\underline{x}),
\end{aligned}$$

which is at zero at $y = a_2\underline{x}$, where the coefficients C_3 and C_1 are given by Lemma B.1 with μ and σ replaced by μ_2 and σ_2. Now consider its derivative with respect to y,

$$B_2\beta_2 y^{\beta_2-1} - C_1\beta_2 y^{\beta_2-1} - C_3\gamma y^{\gamma-1}.$$

It is at zero if and only if

$$B_2\beta_2 y^{\beta_2} = C_1\beta_2 y^{\beta_2} + C_3\gamma y^\gamma = \beta_2 W_Y(y) - \beta_2 C_3 y^\gamma + C_3\gamma y^\gamma.$$

Combining this with the condition stated in (26)-(29), we have $y = a_2\underline{x}$. But $\lim_{y\to 0} H(x,y) - (W_Y(y) - W_X(\underline{x})) = W_X(\underline{x}) > 0$. Therefore

$$H(x,y) - (W_Y(y) - W_X(x)) > 0 \qquad \text{on } M_3. \tag{B.19}$$

Similarly, we have

$$H(x,y) - (W_Y(y) - W_X(x)) > 0 \qquad \text{on } M_5. \tag{B.20}$$

Therefore $H(x,y) \geq W_Y(y) - W_X(x)$ as desired.
Case 2: To show that it is a superharmonic majorant of $e^{-\rho t}(W_Y(y) - W_X(x))$, consider $(x,y) \in M_8$. We have

$$\begin{aligned} & H(x,y) - [W_Y(y) - W_X(x)] \\ = \; & h_4(x) - (W_Y(\bar{y}) - W_X(x)) \\ = \; & B_4 x^{\beta_4} - C_1 x^{\alpha_1} + C_1 x^{\alpha_1} + C_2 x^{\alpha_2} + C_3 x^\gamma - W_Y(\bar{y}), \end{aligned}$$

which is at zero at $x = a_4\bar{y}$, where C_1, C_2, C_3, α_1 and α_2 are as defined in Lemma B.1 with μ and σ replaced by μ_1 and σ_1 respectively. Its derivative is

$$\left[\beta_4 W_Y(\bar{y}) - C_3(\beta_4 - \gamma)x^\gamma\right] x^{-1},$$

which is at zero if and only if $x = a_4\bar{y}$ and is positive for $x < a_4\bar{y}$. Therefore $H(x,y) \geq W_Y(y) - W_X(x)$ on $M_7 \cup M_8$ and

$$H(x,y) > W_Y(y) - W_X(x) \qquad \text{on } M_7. \tag{B.21}$$

Since $e^{-\rho t}H(x,y)$ is a superharmonic majorant of $e^{-\rho t}(W_Y(y) - W_X(x))$,

$$H(x,y) \geq g^*(x,y).$$

On the other hand, denoting $D' = M_1 \cup M_3 \cup M_5$, we have, by Dynkin's formula,

$$H(x,y) = \mathbb{E}^{(x,y)}\left[e^{-\rho\tau_{D'}} H(X_{\tau_{D'}}, Y_{\tau_{D'}})\right] \leq g^*(x,y).$$

Therefore $g^*(x,y) = H(x,y)$.

In the statement of Theorem 1, the set D is called the continuation region. In Theorem 2, the continuation region is defined by using $W_Y(y) -$

$W_X(x)$ and g^*. Now we show that the definition in Theorem 1 is consistent with that in Theorem 2. For this, it is sufficient to show that in D',

$$W_Y(y) - W_X(x) < H(x, y).$$

But this follows immediately from (38), (39) and (40).

Finally $\beta_2 > \gamma$ and β_4 follows from $\rho > \mu_i\gamma + \sigma_i^2\gamma(\gamma-1)/2$ for $i = 1$, 2. ∎

APPENDIX C

PROOFS FOR SECTION 3

To prove the propositions, it is sufficient to prove that the derivatives of

$$\begin{aligned}
a_1 &= \left[\frac{\beta_1\left(\rho - \mu_2\gamma - \frac{\sigma_2^2}{2}\gamma(\gamma-1)\right)}{\left(\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma-1)\right)(\beta_1-\gamma)}\right]^{1/\gamma} \\
a_2 &= \left[\frac{\beta_2\left(\rho - \mu_2\gamma - \frac{\sigma_2^2}{2}\gamma(\gamma-1)\right)}{\rho(\beta_2-\gamma)}\right]^{1/\gamma} \\
a_4 &= \left[\frac{\beta_4\left(\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma-1)\right)}{\rho(\beta_4-\gamma)}\right]^{1/\gamma}
\end{aligned}$$

with respect to the parameters have the appropriate signs.

Let

$$\begin{aligned}
x_1 &= \sqrt{\left[\frac{\sigma_1^2\gamma - \left(\frac{\sigma_1^2}{2} - \mu_1\right) + \left(\frac{\sigma_2^2}{2} - \mu_2\right)}{\sigma_1^2+\sigma_2^2}\right]^2 + \frac{2\left[\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma-1)\right]}{\sigma_1^2+\sigma_2^2}} \\
y_1 &= \frac{\sigma_1^2\gamma - \left(\frac{\sigma_1^2}{2} - \mu_1\right) + \left(\frac{\sigma_2^2}{2} - \mu_2\right)}{\sigma_1^2+\sigma_2^2} \\
x_2 &= \sqrt{\left[\frac{\sigma_2^2\gamma - \left(\frac{\sigma_2^2}{2} - \mu_2\right) + \left(\frac{\sigma_1^2}{2} - \mu_1\right)}{\sigma_1^2+\sigma_2^2}\right]^2 + \frac{2\left[\rho - \mu_2\gamma - \frac{\sigma_2^2}{2}\gamma(\gamma-1)\right]}{\sigma_1^2+\sigma_2^2}} \\
y_2 &= \frac{\sigma_2^2\gamma - \left(\frac{\sigma_2^2}{2} - \mu_2\right) + \left(\frac{\sigma_1^2}{2} - \mu_1\right)}{\sigma_1^2+\sigma_2^2}.
\end{aligned}$$

Then

$$\begin{aligned}
x_1 &> |y_1| \\
x_2 &> |y_2| \\
\beta_1 &= x_1 + y_1 \\
\frac{2\left[\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma-1)\right]}{\sigma_1^2+\sigma_2^2} &= x_1^2 - y_1^2 \\
\frac{2\left[\rho - \mu_2\gamma - \frac{\sigma_2^2}{2}\gamma(\gamma-1)\right]}{\sigma_1^2+\sigma_2^2} &= x_2^2 - y_2^2 \\
\frac{2\left[\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma-1)\right]}{\sigma_1^2+\sigma_2^2} &= x_2^2 - (\gamma - y_2)^2 \\
\frac{2\left[\rho - \mu_2\gamma - \frac{\sigma_2^2}{2}\gamma(\gamma-1)\right]}{\sigma_1^2+\sigma_2^2} &= x_1^2 - (\gamma - y_1)^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left[\sigma_1^2\gamma - \left(\frac{\sigma_1^2}{2} - \mu_1\right) + \left(\frac{\sigma_2^2}{2} - \mu_2\right)\right]^2 + 2(\sigma_1^2+\sigma_2^2)\left[\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma-1)\right] \\
= &\left[\sigma_2^2\gamma - \left(\frac{\sigma_2^2}{2} - \mu_2\right) + \left(\frac{\sigma_1^2}{2} - \mu_1\right)\right]^2 + 2(\sigma_1^2+\sigma_2^2)\left[\rho - \mu_2\gamma - \frac{\sigma_2^2}{2}\gamma(\gamma-1)\right],
\end{aligned}$$

so that

$$x_1 = x_2 \qquad \text{and} \qquad x_1 - y_1 + \gamma = x_2 + y_2 > 0.$$

With these notations and relations, we have

$$\begin{aligned}
\ln a_1 &= \frac{1}{\gamma}\ln\left[\frac{(x_1+y_1)\left(x_1^2 - (\gamma - y_1)^2\right)}{(x_1^2 - y_1^2)(x_1 + y_1 - \gamma)}\right] = \frac{1}{\gamma}\ln\left[1 + \frac{\gamma}{x_1 - y_1}\right] \\
&= \frac{-1}{\gamma}\ln\left[1 + \frac{-\gamma}{x_2 + y_2}\right]. \qquad \text{(C.1)}
\end{aligned}$$

Proof of Proposition 2: Let ϕ denote either μ_1, μ_2, σ_1, σ_2, or ρ. Then

$$\begin{aligned}
\frac{d\ln a_1}{d\phi} &= \frac{-1}{(x_1 - y_1)(x_1 - y_1 + \gamma)}\left[\frac{dx_1}{d\phi} - \frac{dy_1}{d\phi}\right] \\
\frac{d\ln a_1}{d\phi} &= \frac{-1}{(x_2 + y_2)(x_2 + y_2 - \gamma)}\left[\frac{dx_2}{d\phi} + \frac{dy_2}{d\phi}\right].
\end{aligned}$$

Part (a) follows from

$$\frac{dx_2}{d\mu_1} + \frac{dy_2}{d\mu_1} = \left(\frac{y_2}{x_2} + 1\right)\frac{dy_2}{d\mu_1} = \left(1 + \frac{y_2}{x_2}\right)\frac{-1}{\sigma_1^2 + \sigma_2^2} < 0.$$

Part (b) follows from

$$\frac{dx_1}{d\mu_2} - \frac{dy_1}{d\mu_2} = \left(\frac{y_1}{x_1} - 1\right)\frac{dy_1}{d\mu_2} = \left(1 - \frac{y_1}{x_1}\right)\frac{1}{\sigma_1^2 + \sigma_2^2} > 0.$$

For part (c), since

$$\begin{aligned}
\frac{dx_2}{d\sigma_1^2} &= \frac{y_2}{x_2}\frac{dy_2}{d\sigma_1^2} - \frac{\left[\rho - \mu_2\gamma - \frac{\sigma_2^2}{2}\gamma(\gamma-1)\right]}{x_2(\sigma_1^2+\sigma_2^2)^2} \\
\frac{dy_2}{d\sigma_1^2} &= \frac{\frac{1}{2} - y_2}{\sigma_1^2+\sigma_2^2} \\
\frac{dx_2}{d\sigma_1^2} + \frac{dy_2}{d\sigma_1^2} &= \frac{1}{\sigma_1^2+\sigma_2^2}\left[\left(\frac{1}{2} - y_2\right)\left(\frac{y_2}{x_2} + 1\right) - \frac{\left[\rho - \mu_2\gamma - \frac{\sigma_2^2}{2}\gamma(\gamma-1)\right]}{x_2(\sigma_1^2+\sigma_2^2)}\right] \\
&= \frac{1}{(\sigma_1^2+\sigma_2^2)x_2}\left[\left(\frac{1}{2} - y_2\right)(x_2+y_2) - \frac{1}{2}(x_2^2 - y_2^2)\right] \\
&= \frac{-(x_2+y_2)}{2(\sigma_1^2+\sigma_2^2)x_2}\left(x_2+y_2-1\right),
\end{aligned}$$

the sign of $da/d\sigma_1^2$ is the same as that of $x_2 + y_2 - 1$. If $1 - y_2 > 0$, which is equivalent to

$$\mu_2 - \mu_1 < \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}(3 - 2\gamma),$$

then $x_2 + y_2 - 1 > 0$ if and only if

$$\rho > \mu_2(\gamma - 1) + \frac{\sigma_2^2}{2}(1-\gamma)(2-\gamma) + \mu_1.$$

If $1 - y_2 \leq 0$, which is equivalent to

$$\mu_2 - \mu_1 \geq \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}(3 - 2\gamma),$$

then $x_2 + y_2 - 1 > 0$. Part (d) follows from

$$\begin{aligned}
\frac{dx_1}{d\sigma_2^2} &= \frac{y_1}{x_1}\frac{dy_1}{d\sigma_2^2} - \frac{\left[\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma-1)\right]}{x_1(\sigma_1^2+\sigma_2^2)^2} \\
\frac{dy_1}{d\sigma_2^2} &= \frac{\frac{1}{2} - y_1}{\sigma_1^2+\sigma_2^2} \\
\frac{dx_1}{d\sigma_2^2} - \frac{dy_1}{d\sigma_2^2} &= \frac{1}{\sigma_1^2+\sigma_2^2}\left[\left(\frac{1}{2} - y_1\right)\left(\frac{y_1}{x_1} - 1\right) - \frac{\left[\rho - \mu_1\gamma - \frac{\sigma_1^2}{2}\gamma(\gamma-1)\right]}{x_1(\sigma_1^2+\sigma_2^2)}\right] \\
&= \frac{1}{(\sigma_1^2+\sigma_2^2)x_1}\left[\left(y_1 - \frac{1}{2}\right)(x_1 - y_1) - \frac{1}{2}(x_1^2 - y_1^2)\right] \\
&= \frac{-(x_1-y_1)}{2(\sigma_1^2+\sigma_2^2)x_1}(1 + x_1 - y_1) < 0.
\end{aligned}$$

(e) follows from

$$\frac{dx_1}{d\rho} - \frac{dy_1}{d\rho} = \frac{1}{x_1}\frac{1}{\sigma_1^2+\sigma_2^2} < 0. \qquad \blacksquare$$

Let

$$\begin{aligned}
w_1 &= \sqrt{\left(\frac{\mu_1}{\sigma_1^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_1^2}}, & z_1 &= \frac{\mu_1}{\sigma_1^2} - \frac{1}{2} \\
w_2 &= \sqrt{\left(\frac{\mu_2}{\sigma_2^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma_2^2}}, & z_2 &= \frac{\mu_2}{\sigma_2^2} - \frac{1}{2}.
\end{aligned}$$

Then $\beta_2 = w_2 - z_2$, $w_2 > |z_2|$, $\beta_4 = -w_1 - z_1$, $w_1 > |z_1|$ and

$$\begin{aligned}
\rho - \mu_1\gamma - \tfrac{\sigma_1^2}{2}\gamma(\gamma-1) &= \tfrac{\sigma_1^2}{2}\left(w_1^2 - (z_1+\gamma)^2\right) \\
\rho - \mu_2\gamma - \tfrac{\sigma_2^2}{2}\gamma(\gamma-1) &= \tfrac{\sigma_2^2}{2}\left(w_2^2 - (z_2+\gamma)^2\right).
\end{aligned}$$

Thus

$$\ln a_2 = \frac{1}{\gamma}\ln\left[\frac{(w_2 - z_2)\frac{\sigma_2^2}{2}[w_2^2 - (z_2+\gamma)^2]}{\rho(w_2 - z_2 - \gamma)}\right] = \frac{1}{\gamma}\ln\left(1 + \frac{\gamma}{w_2 + z_2}\right), \tag{C.2}$$

and

$$\ln a_4 = \frac{1}{\gamma}\ln\left[\frac{(w_1 + z_1)\frac{\sigma_1^2}{2}[w_1^2 - (z_1+\gamma)^2]}{\rho(w_1 + z_1 + \gamma)}\right] = \frac{1}{\gamma}\ln\left(1 + \frac{-\gamma}{w_1 - z_1}\right). \tag{C.3}$$

Proof of Proposition 1: Parts (a), (b) and (d) are special cases of Proposition 2. Part (c) follows immediately from the expression of upper boundary, i.e., $a_2 x$.

For part (e), first taking the derivative with respect to $w_2 + z_2$, we have

$$\frac{d \ln a_2}{d(w_2 + z_2)} = \frac{-1}{(w_2 + z_2 + \gamma)(w_2 + z_2)}.$$

Then taking the derivative with respect to γ,

$$\frac{d^2 \ln a_2}{d(w_2 + z_2) d\gamma} = \frac{1}{(w_2 + z_2 + \gamma)^2 (w_2 + z_2)} > 0.$$

Thus $d \ln a_2 / d\gamma$ increases in $(w_2 + z_2)$. Since

$$\frac{d \ln a_2}{d\gamma} = \frac{-1}{\gamma^2} \ln\left(1 + \frac{\gamma}{w_2 + z_2}\right) + \frac{1}{\gamma} \frac{1}{w_2 + z_2 + \gamma} \to 0$$

as $w_2 + z_2 \to \infty$, we have that $d \ln a_2 / d\gamma < 0$. ▌

Proof of Proposition 3: Let ϕ denote either μ_1, σ_1, or ρ. Then

$$\frac{d \ln a_4}{d\phi} = \frac{1}{(w_1 - z_1)(w_1 - z_1 - \gamma)} \left[\frac{dw_1}{d\phi} - \frac{dz_1}{d\phi}\right].$$

Note that $\beta_4 < \gamma$ implies that $w_1 - z_1 - \gamma > 0$. Part (a) follows from

$$\left[\frac{dw_1}{d\mu_1} - \frac{dz_1}{d\mu_1}\right] = \left(\frac{z_1}{w_1} - 1\right) \frac{1}{\sigma_1^2} < 0.$$

Part (b) follows from

$$\left[\frac{dw_1}{d\sigma_1^2} - \frac{dz_1}{d\sigma_1^2}\right] = \left(\frac{z_1}{w_1} - 1\right) \frac{-\mu_1}{\sigma_1^4} + \frac{\rho}{w_1} > 0.$$

Part (c) follows from

$$\left[\frac{dw_1}{d\rho} - \frac{dz_1}{d\rho}\right] = \frac{\sigma_1^2}{w_1} > 0.$$

Part (d) can be proved in a similar way as in the proof of Proposition 1 (e). ▌

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