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The Effect of Parameter Uncertainty on Consumption, Wealth, and Welfare

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This paper looks at precautionary saving when income follows a random walk where the trend parameter μ is unknown. Agents use Bayesian econometrics to determine their beliefs about μ. Parameter uncertainty about μ is shown to dominate all other sources of uncertainty that agents face. We derive a closed-form solution to the model and show that this parameter uncertainty has a big impact on consumption, precautionary savings, and welfare. The effect of this parameter uncertainty on wealth is shown to be permanent.

Key Words: Parameter Uncertainty; Bayesian econometrics; Precautionary Savings; Consumption; Wealth.
JEL Classification Numbers: C11, C22, E21.

1. INTRODUCTION

Economists routinely face parameter uncertainty, and so need econometrics when fitting and testing economic models. Agents within these models however are typically assumed not to face parameter uncertainty: they are assumed to know all relevant parameters when making their decisions. If this were literally true there would be no need for econometrics, one could simply use introspection or ask people what the parameters are. Perhaps economists are missing an important source of uncertainty.

In this paper we focus on parameter uncertainty with respect to the slope μ of the trend line of income Y_t, where income Y_t is assumed to follow a random walk with drift. If there were no parameter uncertainty with respect to μ then the conditional variance of future income increases linearly with the forecast horizon. But if μ is unknown the conditional variance grows as the *square* of the forecast horizon (see Sampson (1991)), and hence parameter uncertainty is the dominant source of uncertainty. In the face of this greater uncertainty agents will have a strong incentive to

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save more as insurance against a bad draw of μ. This leads us to believe that parameter uncertainty will have a big impact on consumption, wealth, and welfare.

In this paper we attempt to assess the impact of this parameter uncertainty in a model of precautionary savings. We use a version of the model found in Caballero (1991), but unlike in Caballero we assume μ is unknown. Like Caballero we are able to obtain a closed-form solution to the model when agents act as Bayesian econometricians combining sample and prior information on μ. We show that this parameter uncertainty has a big effects on wealth accumulation and welfare, and that the effects of parameter uncertainty on wealth are permanent.

1.1. The Model

At time t utility is

$$V_t = E_t \sum_{k=0}^{T} U(C_{t+k})e^{-rk},\ r > 0 \tag{1}$$

where $U(C)$ is exhibits constant absolute risk aversion ($CARA$) as

$$U(C) = -\exp(-\theta C)/\theta,\ \theta > 0. \tag{2}$$

Income Y_t follows a random walk with *i.i.d.* normally distributed shocks a_t and drift parameter μ as

$$Y_t = \mu + Y_{t-1} + a_t \text{ where } a_t \sim N\left[0, \sigma^2\right].$$

For income k periods in the future we have

$$Y_{t+k} = Y_t + \mu k + \sum_{j=1}^{k} a_{t+j} \tag{3}$$

so that if μ is known the distribution of Y_{t+k} conditional on Y_t is

$$Y_{t+k} \sim N\left[Y_t + k\mu, \sigma^2 k\right] \tag{4}$$

where the conditional variance $Var_t\left[Y_{t+k}\right] = \sigma^2 k$ grows linearly with the forecast horizon k.

Now let us assume that μ is unknown. (We assume σ is known throughout the paper. In the conclusion we discuss the implications of an unknown σ.) At $t = 0$ the parameter μ is drawn drawn from the distribution

$$\mu \sim N[\hat{\mu}_0, \frac{\sigma^2}{t_0}]. \tag{5}$$

Agents know (5) and so it acts as their prior for μ with prior mean $\hat{\mu}_0$ and prior variance $\frac{\sigma^2}{t_0}$. Because (5) is a conjugate prior, it is equivalent to having t_0 sample observations. If $t_0 = 0$ we have a non-informative prior, while $t_0 = \infty$ reduces to the case where $\mu = \hat{\mu}_0$ is known.

At $t = 0$ agents begin accumulating sample data $D(t)$ as

$$D(t) = [Y_0, Y_1, Y_2, \ldots, Y_t].$$

Since ΔY_t is independent and normally distributed, the posterior for μ comes from standard Bayesian results (see for example Box and Tiao, 1973) and is

$$\mu \sim N\left[\hat{\mu}_t, \frac{\sigma^2}{t + t_o}\right] \tag{6}$$

where the posterior mean $\hat{\mu}_t$ is

$$\hat{\mu}_t = \hat{\mu}_{t-1} + \frac{\Delta Y_t - \hat{\mu}_{t-1}}{t + t_0} = \bar{\mu}_t + t_o \frac{\hat{\mu}_0 - \bar{\mu}_t}{t + t_0} \tag{7}$$

and

$$\bar{\mu}_t = \frac{1}{t} \sum_{j=1}^{t} \Delta Y_j = \frac{Y_t - Y_0}{t}$$

is the sample mean of ΔY_t over $D(t)$. Notice that uncertainty about μ as measured by

$$Var_t[\mu] = \frac{\sigma^2}{t + t_o} \tag{8}$$

falls as the the sum of prior and sample information $t + t_0$ increases. Parameter uncertainty $Var_t[\mu]$ has a maximum value at $t = 0$, and diminishes monotonically thereafter.

The time period $t = 0$ is the beginning of time, not in the sense of the Big Bang, but in the sense of when the current regime begins. For an individual this might be birth or entry into the workforce; for a country this might be when the country emerges from a war, or after a great invention such as electricity or modern computers. The regime change is non-ergodic; that is, data from the old regime $t < 0$ cannot be used to estimate μ in the new regime $t > 0$.

When a regime begins at $t = 0$ parameter uncertainty $Var_t[\mu]$ about μ is at its greatest since there is no sample information. As time progresses uncertainty $Var_t[\mu]$ diminishes and converges to zero as agents collect more and more sample information $D(t)$ about the new regime. Parameter uncertainty itself is thus a *transient* phenomenon. But as we will see, parameter uncertainty has a permanent effect on wealth accumulation.

An unknown μ fundamentally alters uncertainty agents face. To see this note that from (4), (6) and the fact that μ is independent of future shocks that

$$Var_t[Y_{t+k}] = Var_t\left[\mu k + \sum_{j=1}^{k} a_{t+j}\right] = \frac{\sigma^2}{t+t_0}k^2 + \sigma^2 k. \tag{9}$$

The posterior for Y_{t+k} conditional on the information at t is

$$Y_{t+k} \sim N\left[Y_t + k\hat{\mu}_t, \frac{\sigma^2}{t+t_0}k^2 + \sigma^2 k\right]. \tag{10}$$

Comparing (9) or (10) with (4) we see that the conditional variance goes from being $Var_t[Y_{t+k}] = O\left(k\right)$ when μ is known to $Var_t[Y_{t+k}] = O(k^2)$ when μ is unknown.

2. PARAMETER UNCERTAINTY WITHOUT PRECAUTIONARY SAVING

Here we will show that agents will have an overwhelming incentive to prevent the $O\left(k^2\right)$ uncertainty in the income stream Y_t from entering the consumption stream C_t. To do this suppose an agent is prevented from saving so that $C_t = Y_t$. Combining (1) and (10) it follows that welfare V_{1t} at time t is

$$V_{1t} = U(C_t)\sum_{k=0}^{T} \exp\left(-\lambda_{1t}k + \lambda_{2t}k^2\right) \tag{11}$$

where

$$\lambda_{1t} = r + \theta(\hat{\mu}_t - \theta\frac{\sigma^2}{2}) \text{ and } \lambda_{2t} = \frac{\theta^2\sigma^2}{2\left(t+t_0\right)} > 0. \tag{12}$$

The case where μ is known is found by setting $t_0 = \infty$, $\hat{\mu}_t = \mu$ and $\lambda_{1t} = \lambda_1, \lambda_{2t} = 0$ where

$$\lambda_1 = r + \theta(\mu - \theta\frac{\sigma^2}{2}) > 0.$$

Letting $T \to \infty$ we find that

$$\lim_{T\to\infty} V_{1t} = \frac{U(C_t)}{1-\exp\left(-\lambda_1\right)} > -\infty. \tag{13}$$

Now consider the case where μ is unknown. Since the term $\lambda_{2t}k^2 > 0$ in (11) increases as $O\left(k^2\right)$, it must eventually dominate the $O(k)$ term $-\lambda_{1t}k$ for large enough T and so

$$\lim_{T\to\infty} V_{1t} = -\infty. \tag{14}$$

Comparing (13) and (14) it follows so that for a long enough time horizon T, the uncertainty regarding μ dominates welfare.

The welfare cost of uncertainty can be measured by the amount of consumption $\tilde{\gamma}_t(T)$ an agent would be willing to sacrifice in return for knowing that future income will be $Y_t + k\hat{\mu}_t$ with certainty. This is defined by

$$U(C_t)\sum_{k=0}^{T}\exp\left(-\lambda_{1t}k + \lambda_{2t}k^2\right) = U\left(C_t - \tilde{\gamma}_t(T)\right)\sum_{k=0}^{T}\exp\left(-\lambda_{1t}k\right)$$

so that

$$\tilde{\gamma}_t(T) = \frac{1}{\theta}\ln\left[\frac{\sum_{k=0}^{T}\exp\left(-\lambda_{1t}k + \lambda_{2t}k^2\right)}{\sum_{k=0}^{T}\exp\left(-\lambda_{1t}k\right)}\right]. \tag{15}$$

We then see that

$$\lim_{T\to\infty}\tilde{\gamma}_t(T) = \infty \tag{16}$$

so that the welfare costs of an unknown μ increase without bound.

3. PARAMETER UNCERTAINTY WITH PRECAUTIONARY SAVING

Suppose that the agent can borrow and save at a constant real rate of interest r so that wealth A_t evolves as

$$A_t = e^r(A_{t-1} + Y_{t-1} - C_{t-1}),\ r > 0,\ A_0 = 0. \tag{17}$$

There is no bequest motive so that $A_{T+1} = 0$ and

$$C_T = Y_T + A_T. \tag{18}$$

Maximizing welfare at time t leads to the standard Euler equation

$$U'(C_t) = E_t\left[U'(C_{t+1})\right] \tag{19}$$

or

$$\exp(-\theta C_t) = E_t\left[\exp(-\theta C_{t+1})\right]. \tag{20}$$

In the appendix it is shown that

$$C_t = Y_t + \alpha_t(T)A_t + \beta_t(T)\hat{\mu}_t - \gamma_t(T) \tag{21}$$

where

$$\begin{aligned}
\alpha_t(T) &= \frac{1-e^{-r}}{1-e^{-r(T+1-t)}} \qquad (22)\\
\beta_t(T) &= \frac{e^{-r}}{1-e^{-r}}\left[\frac{1-e^{-r(T-t)}-(T-t)\,e^{-r(T-t)}\,(1-e^{-r})}{1-e^{-r(T+1-t)}}\right]\\
\gamma_t(T) &= \frac{\alpha_t(T)\theta\sigma^2}{2}\sum_{k=1}^{T-t}\frac{e^{-rk}}{\alpha_{t+k+1}(T)}\left(1+\frac{\beta_{t+k+1}(T)}{t+t_o+k}\right)^2\left(1+\frac{1}{t+t_o+k-1}\right).
\end{aligned}$$

Saving $S_t \equiv Y_t - C_t$ then is

$$S_t = -\alpha_t(T)A_t - \beta_t(T)\hat{\mu}_t + \gamma_t(T). \tag{23}$$

Now consider what happens if we allow an infinite horizon. Exponential discounting insures that as $T \to \infty$, the limits of $\alpha_t(T), \beta_t(T), \gamma_t(T)$ are all finite and given respectively by

$$\begin{aligned}
\alpha &= 1-e^{-r} \qquad (24)\\
\beta &= \frac{e^{-r}}{1-e^{-r}}\\
\gamma_t &= \frac{\theta\sigma^2}{2}\sum_{k=1}^{\infty}e^{-rk}\left(1+\frac{\beta}{t+t_o+k}\right)^2\left(1+\frac{1}{t+t_o+k-1}\right).
\end{aligned}$$

Using (19) welfare V_{2t} with precautionary savings is finite

$$V_{2t} = U(C_t)\sum_{k=0}^{\infty}e^{-rk} = \frac{U(C_t)}{1-e^{-r}} > -\infty \tag{25}$$

unlike $V_{1t} = -\infty$ in (14) when $C_t = Y_t$. The welfare costs of uncertain future income are finite, even with an infinite horizon.

The infinite horizon $T = \infty$ can only be possible if the agent has shifted the $O(k^2)$ uncertainty in income Y_t to saving S_t, thereby insulating consumption C_t from the $O(k^2)$ uncertainty in Y_t. We now show this explicitly. From the Euler equation in (20), the law of iterated expectations, and the conditional normality of C_{t+k} it follows that

$$\exp(-\theta E_t\left[C_{t+k}\right] + \frac{\theta^2}{2}Var_t[C_{t+k}]) = E_t[\exp(-\theta C_{t+k})] = \exp(-\theta C_t)$$

and so

$$Var_t[C_{t+k}] = \frac{2}{\theta} E_t \left[C_{t+k} - C_t \right]. \tag{26}$$

The conditional variance $Var_t[C_{t+k}]$ grows at the same rate as the conditional mean $E_t \left[C_{t+k} - C_t \right]$. From (21) with $T = \infty$ we have

$$E_t[C_{t+k} - C_t] = k\hat{\mu}_t + (1-e^{-r})E_t[A_{t+k} - A_t] + \gamma_t - \gamma_{t+k} = (e^r - 1)\sum_{j=0}^{k-1} \gamma_{t+j} + \gamma_t - \gamma_{t+k}.$$

Since γ_t is a decreasing function of t we have $Var_t[C_{t+k}] = O\left(k\right)$ as

$$Var_t[C_{t+k}] \leq (k+1)\,\frac{2}{\theta}(e^r - 1)\gamma_t.$$

We now show that the effect of parameter uncertainty on wealth is permanent. This seems surprising since parameter uncertainty itself is transitory since as $t \to \infty$ the sample information $D\left(t\right)$ reveals μ as $\hat{\mu}_t \overset{p}{\to} \mu$ and $Var_t\left[\mu\right] \to 0$.

With $T = \infty$ and (21) we have

$$A_t + S_t = e^{-r} A_t - \beta\hat{\mu}_t + \gamma_t. \tag{27}$$

Since $A_{t+1} = e^r (A_t + S_t)$ we have

$$A_{t+1} = A_t - \frac{\hat{\mu}_t}{1 - e^{-r}} + e^r \gamma_t. \tag{28}$$

Since $A_0 = 0$ we can solve for wealth A_t as

$$A_t = -\frac{1}{1 - e^{-r}} \sum_{k=1}^{t} \hat{\mu}_{t-k} + e^r \sum_{k=1}^{t} \gamma_{t-k}. \tag{29}$$

From (29) it follows that parameter uncertainty k periods in the past, as captured by γ_{t-k}, has a permanent impact on present wealth A_t. This means that the effect of parameter uncertainty via γ_t on wealth A_t is permanent.

We now attempt to assess the relative importance of different sources of uncertainty on wealth accumulation. We can decompose A_t into three terms as

$$A_t = A_{1t} + A_{2t} + A_{3t}.$$

The first of these

$$A_{1t} = -\frac{1}{1 - e^{-r}} \sum_{k=1}^{t} \hat{\mu}_{t-k}$$

is random, reflecting how changing estimates of μ affect wealth, and will depend on the sample $D(t)$.

The terms A_{2t} and A_{3t} are non-random and reflect uncertainty due to unknown future shocks and an unknown μ. The second term A_{2t} is accumulated wealth due to unknown future shocks, and is found by letting $t_0 \to \infty$ in the second term in (29) as

$$A_{2t} = e^r \lim_{t_0 \to \infty} \sum_{k=1}^{\infty} \gamma_{t-k} = \frac{\theta \sigma^2}{2} \frac{e^{-r}}{1 - e^{-r}}. \tag{30}$$

The third term A_{3t} is wealth accumulated because of parameter uncertainty

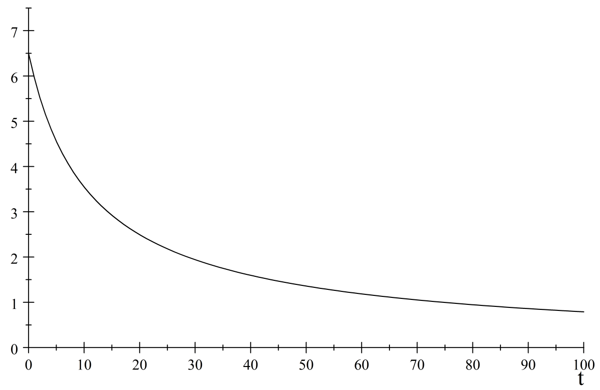
$$A_{3t} = e^r \sum_{k=1}^{t} \gamma_{t-k} - A_{2t}.$$

The relative importance of these two sources of uncertainty for wealth accumulation can then be measured by

$$\begin{aligned} \phi(t) &= \frac{A_{3t} - A_{2t}}{A_{2t}} \\ &= \left(1 - e^{-r}\right) \sum_{k=1}^{\infty} e^{-r(k-1)} \left(1 + \frac{\frac{e^{-r}}{1-e^{-r}}}{t + t_0 + k}\right)^2 \left(1 + \frac{1}{t + t_0 + k - 1}\right) - 1. \end{aligned} \tag{31}$$

Note that $\phi(t)$ does not depend on the degree of risk aversion θ or the variance of shocks σ^2. If we measure time in years and set $t_0 = 10$ and $r = 0.02$ we obtain $\phi(t)$ shown in Figure 1. Examining Figure 1 we see that wealth accumulation due to parameter uncertainty regarding μ is still important even after the new regime has lasted 100 years.

FIG. 1. $\phi(t)$ for $t_0 = 10$ and $r = 0.02$



4. CONCLUSIONS

We have shown that parameter uncertainty with respect to the trend coefficient μ of a random walk for income can have a big impact on consumption, wealth, and welfare. The random walk assumption itself is not critical to this result: any time series process with a trend, either difference or trend stationary, will yield similar results (see Sampson (1991)). Qualitatively different results however can be expected if one relaxes the assumption that the standard deviation of shocks σ is known. This will change the posterior for future income from the thin-tailed normal distribution found in this paper, to a thick-tailed distribution like the t distribution. These thicker tails caused by an unknown σ can easily lead to an even greater impact than the unknown μ considered in this paper.

APPENDIX

From the Euler equation (20) for $t-1$ substitute (21) in for C_t to yield

$$\begin{aligned} \exp(-\theta C_{t-1}) &= E_{t-1}\left[\exp\left(-\theta(Y_t + \alpha_t A_t + \beta_t \hat{\mu}_t - \gamma_t)\right)\right] \qquad (A.1)\\ &= E_{t-1}\left[\exp\begin{pmatrix} -\theta(Y_{t-1} + \Delta Y_t + \alpha_t e^r\left(A_{t-1} + Y_{t-1} - C_{t-1}\right) \\ +\beta_t\left(\hat{\mu}_{t-1} + \frac{\Delta Y_t - \hat{\mu}_{t-1}}{t+t_0}\right) - \gamma_t) \end{pmatrix}\right] \end{aligned}$$

where the second equality follows from $Y_t = Y_{t-1} + \Delta Y_t$, (7) and (17). Collecting the terms which are in the information set and those which are not, we find that the term not in the information set will be

$$E_{t-1}\left[\exp\left(-\theta\left(1+\frac{\beta_t}{t+t_0}\right)\right)\Delta Y_t\right] = \exp\left[\begin{array}{c} \left(-\theta\left(1+\frac{\beta_t}{t+t_0}\right)\right)\hat{\mu}_{t-1} \\ +\frac{\theta^2\sigma^2}{2}\left(\left(1+\frac{\beta_t}{t+t_0}\right)^2\left(1+\frac{1}{t+t_0-1}\right)\right) \end{array}\right] \qquad (A.2)$$

since the distribution of ΔY_t conditional on the information set at $t-1$ is

$$\Delta Y_t \sim N\left[\hat{\mu}_{t-1}, \sigma^2\left(1+\frac{1}{t+t_0-1}\right)\right]$$

which follows from (10).

Using (A.2) in (A.1) and solving for C_{t-1} then yields

$$C_{t-1} = Y_{t-1} + \alpha_{t-1}A_{t-1} + \beta_{t-1}\hat{\mu}_{t-1} - \gamma_{t-1} \qquad (A.3)$$

where

$$\alpha_{t-1} = \frac{\alpha_t e^r}{1+\alpha_t e^r}, \beta_{t-1} = \frac{1+\beta_t}{1+\alpha_t e^r}, \text{ and}$$
$$\gamma_{t-1} = \frac{\gamma_t + \frac{\theta^2\sigma^2}{2}\left(\left(1+\frac{\beta_t}{t+t_0}\right)^2\left(1+\frac{1}{t+t_0-1}\right)\right)}{1+\alpha_t e^r}. \tag{A.4}$$

From (A.4) it follows that

$$\alpha_{t-1}^{-1} = e^{-r}\alpha_t^{-1} + 1 \tag{A.5}$$

so that using $\alpha_T = 1$ and solving (A.5) forwards results in the first equation in (22). To obtain the remaining required results note that from (A.4)

$$\frac{1}{1+\alpha_t e^r} = \frac{e^{-r}\alpha_{t-1}}{\alpha_t} \tag{A.6}$$

so that if $\tilde{\beta}_t = \frac{\beta_t}{\alpha_t}$ and $\tilde{\gamma}_t = \frac{\gamma_t}{\alpha_t}$ then β_t and γ_t in (A.4) can be rewritten as

$$\tilde{\beta}_{t-1} = e^{-r}\tilde{\beta}_t + \frac{e^{-r}}{\alpha_t}, \tilde{\gamma}_{t-1}$$
$$= e^{-r}\tilde{\gamma}_t + \frac{e^{-r}}{\alpha_t}\frac{\theta^2\sigma^2}{2}\left(\left(1+\frac{\beta_t}{t+t_0}\right)^2\left(1+\frac{1}{t+t_0-1}\right)\right). \tag{A.7}$$

Again solving forwards and using $\tilde{\beta}_T = \tilde{\gamma}_T = 0$ then yields, after some straightforward manipulation, the required results.

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