Dynamic Aggregation and Computation of Equilibria in Finite-Dimensional Economies with Incomplete Financial Markets [*]

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This paper constructs a representative agent supporting the equilibrium allocation in "event-tree" economies with time-additive preferences and possibly incomplete securities markets. If the equilibrium allocation is Pareto optimal, this construction gives the usual linear welfare function. Otherwise, the representative agent's utility function is state-dependent, even when individual agents have state-independent utilities and homogeneous beliefs. The existence of a representative agent allows us to provide a characterization of equilibria which does not rely on the derivation of the agents' intertemporal demand functions for consumption and investment and transforms the dynamic general equilibrium problem into a static one. This characterization is therefore especially well suited for numerical computation of equilibria in economies with incomplete financial markets. © 2001 Peking University Press

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1. INTRODUCTION

It is well known that competitive equilibria in economies with complete financial markets can be supported by a "representative agent", in the sense that it is always possible to replace the original consumers/investors with a single agent who acts as a proxy for the individual consumers by receiving the aggregate endowment and optimally holding the total supply of securities (and consuming the aggregate income) at the competitive equilibrium prices. Moreover, the utility function of the representative agent corresponds to a linear social welfare function (i.e., to a weighted sum of utilities), which is completely identified by the weightings received by the individual utilities (see Negishi (1960), Constantinides (1982) and Huang (1987)). The search for an equilibrium can therefore be reduced to the search for such weightings, as the equilibrium price process and consumption allocation can be easily recovered from the representative agent's utility function. Negishi (1960) has first used this approach to provide an alternative proof of the existence of equilibria in Arrow-Debreu economies under the standard neoclassical assumptions on preferences. Karatzas, Lehoczky and Shreve (1990) and Karatzas, Lakner, Lehoczky and Shreve (1991) have recently exploited a similar technique to derive conditions for existence and uniqueness of equilibrium in financial economies with complete securities markets and continuous trading. Similarly, Huang (1987) and Dumas (1989), among others, have utilized the representative agent approach to characterize properties of equilibrium asset prices in models with complete securities markets and continuous trading.

In this paper we show that the same type of analysis can be used to characterize equilibria in event-tree economies with time-additive preferences and incomplete financial markets. Of course, equilibrium allocations with incomplete financial markets are typically Pareto-inefficient, so that a linear welfare function with constant weights will in general not exist. Nevertheless, we show that the construction of a representative agent is still possible: the representative agent's utility function will again be a linear combination of the individual utilities, but with *stochastic weights*, which correspond to the equilibrium marginal rates of substitution across individuals.[1] Since the equilibrium allocation and prices can be characterized entirely in terms of these weights, the task of solving for an equilibrium can again be reduced to the task of searching for the weightings in the representative agent's utility that support the equilibrium allocation.[2] As shown

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[1]Constantinides (1982; p. 255) has observed that the construction of a representative agent with state-dependent utilities should be possible with incomplete markets, although he does not pursue this possibility.

[2]The same approach has been used by Conze, Lasry and Scheinkman (1993) in an international trade model with borrowing constraints.

in the paper, the latter task boils down to solving a system of non-linear algebraic equations. The advantage of this characterization is that it does not require the derivation of the agents' intertemporal demand functions for consumption and investment and it allows us to transform the dynamic general equilibrium problem into an equivalent static one: it is therefore especially well suited for numerical computation. Moreover, since the dimension of the vector of weightings depends on the number of agents in the economy and on the structure of uncertainty, but not on the number of traded securities, our characterization of equilibria is typically more parsimonious than characterizations based on the stochastic Euler equations from dynamic programming.

In constructing a representative agent and deriving the conditions characterizing an equilibrium, we use an approach similar to the martingale representation technique introduced by Pliska (1986), Cox and Huang (1989, 1991), and Karatzas, Lehoczky and Shreve (1987) in the case of complete markets. This approach proceeds in three steps: first the set of feasible consumption processes is characterized, then the agents' optimal consumption processes in this set are determined using Lagrangian theory, and finally the trading strategies needed to generate the optimal consumption process are derived. With complete markets, the only constraint placed on individual consumption policies is budget-feasibility: therefore, the set of feasible consumption processes corresponds to the set of consumption processes that are budget-feasible with respect to the unique set of underlying state prices. With incomplete markets, the characterization of feasible consumption processes is more complex, since there is in general more than one set of state prices consistent with any given equilibrium, and not every budget-feasible consumption process is attainable through a dynamic trading strategy. He and Pearson (1991) have recently shown that, with discrete trading, the set of consumption bundles that can be generated by a budget-feasible dynamic trading strategy in incomplete markets is identical to the set of consumption bundles that are budget feasible with respect to *all* of the state prices consistent with no arbitrage and can be characterized in terms of a finite number of budget constraints corresponding to the extreme points of the closure of that set.[3] We use a similar characterization of feasible consumption processes.

Building on the martingale representation technique, the representative agent approach developed in this paper allows us to transform a dynamic general equilibrium problem into a static one in which the search for equilibrium is reduced to the search for the stochastic weights (across time

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[3]An essentially equivalent characterization of the feasible consumption policies as those having the same price under all the equivalent martingale measures (state price densities) consistent with the given price process has been provided by Pagès (1987) and Jacka (1992) for infinite-dimensional economies.

and across agents) that define the representative agent's utility function. In turn, the latter task involves solving a system of non-linear algebraic equations. In contrast to the standard approach in the literature, dynamic programming is not required in this process.

There is by now a vast literature in mathematical economics analyzing the existence of general equilibrium in models with incomplete markets (see the review articles by Geanakoplos (1990) and Magill and Shafer (1991), and the references therein). While this literature typically allows for multiple goods and general utility functions, we assume that agents consume a single numeraire good and that utility functions are time-additive. These assumptions, which are standard in the finance literature, are critical for our approach.

A homotopy technique for the computation of equilibria in incomplete markets has recently been proposed by Brown, DeMarzo and Eaves (1996). However, they assume that the economy is specified directly in terms of an excess demand function, rather than of individual utilities and endowments. The main focus of the approach proposed in this paper is precisely on avoiding the computation of the demand functions.

The balance of the paper is organized as follows. Section 2 introduces the main ideas through a simple example with complete markets. Section 3 starts our analysis with incomplete markets by describing the economy. Section 4 formulates the static individual optimization problem and the notion of equilibrium. Section 5 constructs the representative agent and provides a characterization of the weightings in the representative agent's utility function which lead to an equilibrium. Section 6 uses this result to show how the search for an equilibrium can be reduced to the search of the weightings defining the representative agent's utility function. Section 7 shows how this approach simplifies in the case in which the market structure consists of a sequence of one-period securities. Section 8 contains some concluding remarks.

2. AN EXAMPLE WITH COMPLETE MARKETS

We introduce the basic computational approach of the paper via a simple example with complete financial markets. Consider an economy with a single consumption good (the numeraire), two dates ($t = 0, 1$), and two possible states (ω_1, ω_2) at date 1, having equal probability ($\pi(\omega_1) = \pi(\omega_2) = \frac{1}{2}$). It is convenient to consider date 0 as just an additional state by redefining the set of "states" to be $\Xi = (\xi_0, \xi_1, \xi_2)$ with $\pi(\xi_0) = 1$ and $\pi(\xi_1) = \pi(\xi_2) = \frac{1}{2}$. Here "state" ξ_0 corresponds to date 0, "state" ξ_1 to state ω_1 at date 1, and "state" ξ_2 to state ω_2 at date 1. We assume complete markets with Arrow-Debreu securities: i.e., there are two securities, the first of which pays one unit of the consumption good at date 1 in state ω_1 and nothing

otherwise, and the second of which pays one unit of the consumption good at date 1 in state ω_2 and nothing otherwise. These securities are in zero net supply. Following our convention of considering date 0 as just an additional state, we can represent the two securities by their dividend processes $d_1 = (0, 1, 0)$ and $d_2 = (0, 0, 1)$. We will also assume that there are two agents with utilities

$$U_1(c) = \sum_{\xi \in \Xi} \pi(\xi) \log c(\xi) \qquad \text{and} \qquad U_2(c) = -\sum_{\xi \in \Xi} \pi(\xi) c(\xi)^{-1}.$$

The first agent is endowed with one unit of the consumption good at date 0 and one unit of the consumption good at date 1 in each possible state, while the second agent is endowed with one unit of the consumption good at date 0, one unit of the consumption good at date 1 in state ω_1 and two units of the consumption good at date 1 in state ω_2. We will write the endowment processes for the two agents as $e_1 = (1, 1, 1)$ and $e_2 = (1, 1, 2)$, respectively. The aggregate endowment process is then given by $\bar{e} = e_1 + e_2 = (2, 2, 3)$.

Let $S = (S_1, S_2)$ denote the prices at date 0 of the two securities. It is well known that if (S, c_1^*, c_2^*) is an equilibrium for the above economy, then there exists a $\lambda^* > 0$ such that $(S, \bar{e})$ is a no-trade equilibrium for the economy with a single representative agent with utility

$$U(c) = \sum_{\xi \in \Xi} \pi(\xi) u(c(\xi)) \tag{1}$$

where

$$u(c) = \max_{z_1 + z_2 = c} \log z_1 - \lambda^* z_2^{-1}. \tag{2}$$

Moreover

$$u(\bar{e}(\xi)) = \log c_1^*(\xi) - \lambda^* c_2^*(\xi) \tag{3}$$

i.e., $(c_1^*(\xi), c_2^*(\xi))$ solves the problem in (2) when $c = \bar{e}(\xi)$. This implies that

$$c_1^*(\xi) = \frac{2\bar{e}(\xi)^2}{\lambda^* + 2\bar{e}(\xi) + \sqrt{\lambda^*(\lambda^* + 4\bar{e}(\xi))}} \tag{4}$$

and

$$c_2^*(\xi) = \left(\frac{2\lambda^* \bar{e}(\xi)^2}{\lambda^* + 2\bar{e}(\xi) + \sqrt{\lambda^*(\lambda^* + 4\bar{e}(\xi))}} \right)^{1/2}. \tag{5}$$

Note that (4) and (5) relate the individual optimal consumption policies in "state" ξ to the aggregate consumption in that "state": these expressions are simply the Pareto-optimal sharing rules. Also, since $\bar{e}$ is an optimal

consumption choice for the representative agent given the price process S, the stochastic Euler equations give:

$$S_1 = \pi(\xi_1) \frac{u'(\bar{e}(\xi_1))}{u'(\bar{e}(\xi_0))} = \frac{1}{2} \tag{6}$$

$$S_2 = \pi(\xi_2) \frac{u'(\bar{e}(\xi_2))}{u'(\bar{e}(\xi_0))} = \frac{2(6 + \lambda^* + \sqrt{\lambda^*(\lambda^* + 12)})}{9(4 + \lambda^* + \sqrt{\lambda^*(\lambda^* + 8)})}. \tag{7}$$

Equations (4)–(7) show that in order to find the equilibrium price and consumption policies (S, c_1^*, c_2^*), we only need to find the weight λ^* that defines the representative agent utility. To determine λ^*, observe that the budget constraint for agent 1 can be written as

$$(c_1^*(\xi_0) - 1) + (c_1^*(\xi_1) - 1)S_1 + (c_1^*(\xi_2) - 1)S_2 = 0 \tag{8}$$

(we will show later that the above equation is in fact a necessary and sufficient condition for λ^* to define an equilibrium). Substituting the expressions (4), (6) and (7) in (8) and solving for λ^* gives a unique (real) solution, $\lambda^* = 1.366$, which implies $S = (0.500, 0.289)$, $c_1^* = (0.895, 0.895, 1.547)$ and $c_2^* = (1.105, 1.105, 1.453)$.

This is essentially the approach used by Negishi (1960) in a single period model and by Dumas (1989) and Karatzas *et alii* (1990, 1991) in a multi-period setting with continuous trading. In the remainder of the paper, we will generalize this approach to incomplete securities markets with discrete trading. More specifically, we will show that the same approach is still valid provided we are willing to allow λ^* to be a stochastic process.

3. THE ECONOMY

We consider a finite-horizon, pure-exchange economy with discrete trading, modeled as follows.

Information structure. There is a finite number of trading dates, indexed by $t = 0, 1, \ldots, T$, with $T \geq 1$, and a finite number of possible states of the world, indexed by $\omega \in \Omega$. As usual, a state of the world is taken to be a complete description of the exogenous relevant uncertainty from time 0 to time T. Information about the true state is partially revealed to individuals over time and is represented by a refining sequence $F = \{F_t : t = 0, 1, \ldots, T\}$ of partitions of Ω, such that $F_0 = \Omega$ and $F_T = \{\{\omega\} : \omega \in \Omega\}$ is the discrete partition. We will refer to F as the *information structure*. Also, we let $\mathcal{F}_t = \sigma\{F_t\}$ be the σ-field generated by F_t and $\mathcal{F} = \{\mathcal{F}_t : t = 0, 1, \ldots, T\}$ be the corresponding filtration.

The above information structure can be usefully visualized as an event tree, as follows. A pair $\xi = (t, a_t)$ with $t \in \{0, 1, \ldots, T\}$ and $a_t \in F_t$ will

be referred to as a *node*. Let

$$\Xi = \bigcup_{\substack{t\in\{0,1,\ldots,T\} \\ a_t\in F_t}} (t, a_t)$$

denote the set of nodes. The node $\xi_0 = (0, \Omega)$ is called the *initial node*. Let $\Xi^+ = \Xi \setminus \xi_0$ denote the set of non-initial nodes. For each $\xi = (t, a_t) \in \Xi^+$, the unique node $\xi^- = (t-1, a_{t-1}) \in \Xi$ with $a_{t-1} \supset a_t$ is called the *predecessor* of ξ. A node ξ with $\xi = (T, \omega)$ is called a *terminal node*. Let Ξ_T denote the set of terminal nodes and $\Xi^- = \Xi \setminus \Xi_T$ denote the set of non-terminal nodes. For each $\xi = (t, a_t) \in \Xi^-$, the set

$$\xi^+ = \{\hat\xi \in \Xi : \hat\xi = (t+1, a_{t+1}), \ a_{t+1} \subset a_t\}$$

is called the set of *immediate successors* of ξ. The number of elements in ξ^+ is called the *branching number* of the node ξ and will be denoted by $b(\xi)$. We say that the node $\hat\xi = (\tau, a_\tau)$ *succeeds* $\xi = (t, a_t)$ if $\tau > t$ and $a_\tau \subset a_t$, and we write $\hat\xi \succ \xi$.

Letting $N = \#\Xi$ denote the number of nodes in Ξ, it is clear that every $\mathbb{R}^m$-valued stochastic process adapted to the filtration $\mathcal{F}$ can be identified with a function from Ξ into $\mathbb{R}^m$, or equivalently with a vector in $\mathbb{R}^{mN}$. In the following we will use the term *m-dimensional process* to refer interchangeably to a $\mathbb{R}^m$-valued function on Ξ or to a vector in $\mathbb{R}^{mN}$. Also, whenever the latter representation of an adapted process is used, the first m coordinates represent the value at date 0, the next $mb(\xi_0)$ coordinates the possible values at time 1, and so on.

Consumption sets. There is a single perishable good available for consumption at each trading date, and we take this consumption good as the numeraire. Each agent's consumption set is the set of real-valued, strictly positive, adapted processes and is thus denoted by $\mathbb{R}^N_{++}$.[4]

Securities market. The securities market is composed of K long-lived securities, indexed by $k = 1, \ldots, K$, available for trading at each date t.[5] Each security is identified by an adapted dividend process $d_k \in D = \{0\} \times \mathbb{R}^{N-1}$ (expressed in units of the consumption good): negative dividends are not ruled out. We let $S = (S_1, \ldots, S_K)$ denote the ex-dividend price

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[4]For any positive integer n, $\mathbb{R}^n_+$ denotes the positive cone of $\mathbb{R}^n$ (i.e., the set of vectors in $\mathbb{R}^n$ with non-negative components) and $\mathbb{R}^n_{++}$ denotes the interior of $\mathbb{R}^n_+$ (i.e., the set of vectors with strictly positive components). For an n-dimensional vector x, we use the following notation: $x \geq 0 \iff x \in \mathbb{R}^n_+$, $x > 0 \iff x \in \mathbb{R}^n_+ \setminus \{0\}$ and $x \gg 0 \iff x \in \mathbb{R}^n_{++}$.

[5]With a slight complication of notation, the case of securities issued at subsequent nodes could also be treated. In Section 7 we examine the case in which the market structure consists of a sequence of one-period securities.

process, which is to be determined endogenously in equilibrium. Since equilibrium prices are assumed to be observed by agents before trading, the process S is adapted. Also, since prices are ex-dividend, $S(\xi) = 0$ for all $\xi \in \Xi_T$.

Trading strategies. A trading strategy is an adapted (predictable) K-dimensional process $\theta = (\theta_1, \ldots, \theta_K)$, where $\theta_k(\xi)$ denotes the number of shares of security k in the portfolio established at node ξ and carried into the nodes in ξ^+. Since we assume that trading occurs at the ex-dividend prices, there is clearly no point in trading securities at the terminal nodes. Hence, we restrict θ to Ξ^-. The space of trading strategies is therefore $\Theta = \mathbb{R}^{K(\#\Xi^-)}$.

Agents. The economy is populated by a finite number I of agents, indexed by $i = 1, \ldots, I$, with $I \geq 2$. Each individual i is endowed with an adapted income process e_i. There is no initial endowment of securities.[6] We will let $\bar{e} = \sum_{i=1}^{I} e_i$ denote the aggregate endowment process.

ASSUMPTION 1. *The endowment vectors satisfy $e_i \gg 0$ for $i = 1, \ldots, I$.*

The beliefs of agent i are represented by a strictly positive probability measure π_i on $(\Omega, \mathcal{F}_T)$. We will interpret π_i as a function on Ξ by letting $\pi_i(\xi) = \pi_i(a_t) = \sum_{\omega \in a_t} \pi_i(\omega)$ for $\xi = (t, a_t)$.

ASSUMPTION 2. *Each agent i has preferences that can be represented by a time- and state-additive utility function $U_i : \mathbb{R}^N_{++} \to \mathbb{R}$, where*

$$U_i(c) = \sum_{\xi \in \Xi} \pi_i(\xi) u_i(c(\xi), \xi). \tag{9}$$

Since we are allowing the utility function in (9) to be state-dependent, we can assume without loss of generality (by setting $\pi = \pi_1$ and redefining $u_i(\cdot, \xi)$ to be $\frac{\pi_i(\xi)}{\pi(\xi)} u_i(\cdot, \xi)$) that $\pi_i = \pi$ for all i, and we will do so to simplify the notation. We will make the following additional assumption about investors' preferences.

ASSUMPTION 3. *The functions $u_i(\cdot, \xi)$ in (9) are increasing, strictly concave and twice continuously differentiable on $\mathbb{R}_{++}$ for all $i = 1, \ldots, I$ and $\xi \in \Xi$. Moreover, they satisfy the Inada conditions:*

$$\lim_{c \downarrow 0} \frac{\partial}{\partial c} u_i(c, \xi) = \infty \qquad and \qquad \lim_{c \uparrow \infty} \frac{\partial}{\partial c} u_i(c, \xi) = 0.$$

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[6] This is without loss of generality: if agent i has a positive endowment $\bar{\theta}_i$ of shares at time 0 and we denote with $\hat{\theta}_i$ his trading strategy, then letting $e_i(\xi) = \bar{\theta}_i \cdot d(\xi)$ and $\theta_i(\xi) = \hat{\theta}_i(\xi) - \bar{\theta}_i$ reduces to the case considered above.

Assumption 3 guarantees in particular that $\frac{\partial}{\partial c} u_i(c, \xi)$ has a well defined strictly decreasing inverse function $f_i(\cdot, \xi)$ mapping $(0, \infty)$ onto itself and that the solution of the individual maximization problem is an interior solution.

In the following, we will denote with

$$\mathcal{E} = \left((\Xi, \pi), \{U_i, e_i\}_{i=1}^I, d \right) \tag{10}$$

the primitives for the above economy.

4. INDIVIDUALS' OPTIMIZATION PROBLEMS AND EQUILIBRIUM

Each individual i chooses an adapted consumption process c_i and an adapted trading strategy $\theta_i = (\theta_i^1, \ldots, \theta_i^K)$ subject to $c_i \geq 0$,

$$c_i(\xi) - e_i(\xi) + \theta_i(\xi) \cdot S(\xi) \leq \theta_i(\xi^-) \cdot (S(\xi) + d(\xi)) \qquad \forall \xi \in \Xi^- \tag{11}$$

and

$$c_i(\xi) - e_i(\xi) \leq \theta_i(\xi^-) \cdot d(\xi) \qquad \forall \xi \in \Xi_T \tag{12}$$

where $\theta_i(\xi_0^-) \stackrel{\text{def}}{=} 0$ and we have assumed that trading occurs at the ex-dividend prices. A trading strategy θ_i satisfying (11)–(12) is said to *finance* the net trade $c_i - e_i$.

DEFINITION 4.1.   A consumption process c is *feasible* for the endowment e given the price process S if it is adapted, strictly positive, and there exists an adapted portfolio strategy $\theta \in \Theta$ that finances the net trade $c - e$.

Let $B(e, S)$ denote the set of feasible consumption processes for the endowment e given the price S. We then have the following definition of equilibrium.

DEFINITION 4.2.   A (rational expectations) *equilibrium* for the economy $\mathcal{E}$ is a price process S for the long-lived securities and a set $(c^*, \theta^*) = \{(c_i^*, \theta_i^*) : i = 1, \ldots, I\}$ of consumption plans and portfolio strategies such that:

(*i*) c_i^* maximizes $U_i(c)$ on $B(e_i, S)$ for $i = 1, \ldots, I$,
(*ii*) $\theta_i^* \in \Theta$ finances c_i^* for $i = 1, \ldots, I$,
(*iii*) the markets for the consumption good and the securities clear: $\sum_{i=1}^I c_i^* = \bar{e}$, and $\sum_{i=1}^I \theta_i^* = 0$.

While under the usual Arrow-Debreu preference assumptions an equilibrium always exists in economies with dividend processes satisfying a spanning condition that guarantees market completion,[7] existence is more problematic in economies with incomplete markets and real securities (i.e., securities with dividends specified in units of (multiple) consumption goods).[8] Hart (1975) provided an example of a simple economy with two dates and two consumption goods for which no equilibrium exists, and Magill and Quinzii (1996; Proposition 24.1) have a similar non-existence result for a three-dates economy with only one consumption good: in both cases, the failure of the usual existence arguments is due to the discontinuities in the agents' demands at points where changes in prices induce changes in the dimension of the space of marketed consumption processes. Duffie and Shafer (1985, 1986) have however shown that in finite-dimensional economies with incomplete markets and an arbitrary number of consumption goods, an equilibrium exists *generically* (i.e., for all but an exceptional set of economies parameterized by endowments and dividend processes). The following proposition summarizes the existence results for our economy.

PROPOSITION 1 (Existence of equilibrium).   *Under the stated assumptions, the following holds:*

(a) If $T = 1$, the economy $\mathcal{E}$ always has an equilibrium.

(b) If $T \geq 2$, there is an open set $A \subset \mathbb{R}_{++}^{IN} \times D^K$ with null complement such that for each $((e_1, \ldots, e_I), d) \in A$ the economy $((\Xi, \pi), \{U_i, e_i\}_{i=1}^I, d)$ has an equilibrium.

Proof.   The first result was proved by Geanakoplos and Polemarchakis (1986). For the second, it follows from Theorem 1 in Duffie and Shafer (1986) that under the given assumptions there is an open set $A \subset \mathbb{R}_{++}^{IN} \times D^K$ with null complement such that for each $((e_1, \ldots, e_I), d) \in A$ there exists a security-spot market equilibrium, i.e., a price process S^* for the K securities, a price process $p \in \mathbb{R}_+^N$ for the consumption good, and consumption and investment strategies $\{(c_i^*, \theta_i^*)\}_{i=1}^I$ such that (c_i^*, θ_i^*) maximizes U_i sub-

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[7]Letting $D_k(T)$ be the random variable denoting the cumulative dividend paid by security k up to the terminal date T, the required spanning condition is that the processes $M_k(t) = \mathrm{E}[D_k(T)|\mathcal{F}_t]$ form a martingale generator (under some equivalent probability measure). In event-tree economies, this condition requires in particular that the number of securities be no less than the maximum branching number: see Duffie and Huang (1985), Duffie (1986) and Duffie and Zame (1989).

[8]Duffie (1987), extending previous work by Cass (1984) and Werner (1985), has shown that an equilibrium always exists in event-tree economies with incomplete markets and an arbitrary number of consumption goods, provided that *securities are purely financial* (i.e., that dividends are specified in units of account).

ject to

$$p(\xi)(c_i^*(\xi) - e_i(\xi)) + \theta_i^*(\xi) \cdot S^*(\xi) \leq \theta_i^*(\xi^-) \cdot (S^*(\xi) + p(\xi)d(\xi)) \quad \forall \xi \in \Xi^-$$
$$p(\xi)(c_i^*(\xi) - e_i(\xi)) \leq \theta_i^*(\xi^-) \cdot (p(\xi)d(\xi)) \quad \forall \xi \in \Xi_T$$

and the markets clear. Define the price process S by $S(\xi) = S^*(\xi)/p(\xi)$. It is then immediate to see that (S, c^*, θ^*) is an equilibrium for the economy $((\Xi, \pi), \{U_i, e_i\}_{i=1}^I, d)$. ∎

A key result for what follows is that the dynamic problem of maximizing (9) subject to the sequence of dynamic budget constraints in (11)–(12) is equivalent to a static problem. To formulate this problem, we will need an alternative characterization of feasible consumption processes. For a given securities price process S and a non-terminal node $\xi \in \Xi^-$, let $X_S^{\xi,k}$ be the process describing the cash flows from a one-period investment in asset k at node ξ, i.e.,

$$X_S^{\xi,k}(\hat{\xi}) = \begin{cases} -S_k(\xi), & \text{if } \hat{\xi} = \xi \\ S_k(\hat{\xi}) + d_k(\hat{\xi}), & \text{if } \hat{\xi} \in \xi^+ \\ 0, & \text{otherwise.} \end{cases}$$

Let X_S be the $N \times K(\#\Xi^-)$ matrix obtained by putting side by side the vectors $X_S^{\xi,k}$, with $\xi \in \Xi^-$ and $k = 1, \ldots, K$. By the monotonicity of preferences, the budget constraint in (11)–(12) is satisfied as an equality, and hence can be written compactly using the matrix X_S as

$$c_i - e_i = X_S \theta_i,$$

where we have used the convention of considering vectors as column vectors whenever matrix operations are involved. In other words, a consumption process $c_i \in \mathbb{R}_{++}^N$ is feasible for endowment e_i given the price process S if and only if $c_i - e_i \in \text{span}(X_S)$, where $\text{span}(X_S) \subset \mathbb{R}^N$ denotes the subspace spanned by the columns of X_S.

Equivalently, letting

$$\Phi(S) = \{\phi \in \mathbb{R}_{++}^N : \phi^\top X_S = 0, \ \phi(\xi_0) = 1\}$$

denote the set of *state prices* consistent with the given price process S, the absence of arbitrage opportunities[9] implies that

$$\text{span}\Phi(S) = \{\phi \in \mathbb{R}^N : \phi^\top X_S = 0\} = X_S \perp$$

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[9]An *arbitrage opportunity* (free lunch) is a trading strategy θ with $X_S\theta > 0$.

where $X_S \perp$ denotes the orthogonal complement of X_S,[10] and hence

$$c - e \in \text{span}(X_S) \quad \Longleftrightarrow \quad \phi \cdot (c - e) = 0 \quad \forall \phi \in \Phi(S). \qquad (13)$$

Equation (13) shows that a net trade $c - e$ is feasible if and only if it satisfies infinitely many Arrow-Debreu budget constraints formed using all the possible set of state prices $\phi \in \Phi(S)$ (or, equivalently, using all $\phi \in X_S \perp$). However, since $X_S \perp$ is finite-dimensional and thus has a finite generating subset, the orthogonality condition in (13) only needs to be checked with respect to such a generating subset. Moreover, since absence of arbitrage implies that $X_S \perp = \text{span}\Phi(S)$, it is always be possible to choose this generating subset so as to lie in $\Phi(S)$. Nevertheless, we will find it convenient not to restrict the generating subset to consist of strictly positive vectors. Letting

$$\text{aff}\,\Phi(S) = \{\phi \in \mathbb{R}^N : \phi^\top X_S = 0, \ \phi(\xi_0) = 1\}$$

denote the *affine hull* of $\Phi(S)$, the previous discussion then implies the following.

PROPOSITION 2 (Feasible consumption processes). *Let S be a no-arbitrage price process and let $\hat{\Phi}(S) \subset \text{aff}\,\Phi(S)$ be such that $\text{span}\hat{\Phi}(S) = X_S \perp$. A consumption process $c \in \mathbb{R}^N_{++}$ is feasible for initial endowment e given the price system S (i.e., $c \in B(e, S)$) if and only if*

$$\phi \cdot (c - e) = 0 \qquad \forall \phi \in \hat{\Phi}(S). \qquad (14)$$

We will henceforth refer to any element $\phi \in \text{aff}\,\Phi(S)$ as a *pseudo state price*. Of course, with (dynamically) complete markets[11] $\#(\text{aff}\,\Phi(S)) = \dim X_S \perp = 1$, so that (14) involves a unique budget constraint. With incomplete markets, He and Pearson (1991) have provided a similar characterization of feasible consumption processes, using the set $\Phi^e(S)$ of extreme points of the closure of $\Phi(S)$ in $\mathbb{R}^N$ in place of our set $\hat{\Phi}(S)$. While the former characterization is more intuitive, we have chosen the latter since

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[10]Since $\phi \in \text{span}\Phi(S)$ implies $\phi^\top X_S = 0$, clearly $\text{span}\Phi(S) \subset X_S \perp$. To prove the equality of the two linear spaces, it is then enough to prove that $\dim(\text{span}\Phi(S)) \geq \dim X_S \perp$. Let $\{x_1, \ldots, x_R\} \subset X_S \perp$, where $R = \dim X_S \perp$, be any maximal set of linearly independent vectors from $X_S \perp$. Absence of arbitrage implies that $X_S \perp \cap \mathbb{R}^N_{++} \neq \emptyset$, so that at least one of the x_i, say x_1, has a non-zero first component $x_1(\xi_0)$. Define $\hat{x}_i = x_i/x_i(\xi_0)$ if $x_i(\xi_0) \neq 0$ and $\hat{x}_i = (x_i + x_1)/x_1(\xi_0)$ otherwise. Then $\{\hat{x}_1, \ldots, \hat{x}_R\}$ is a set of $\dim X_S \perp$ linearly independent vectors from $\Phi(S)$. Hence, $\dim(\text{span}\Phi(S)) \geq R$.

[11]The market is said to be *dynamically complete* if $\dim(\text{span}X_S) = N - 1$.

it is more general (see Girotto and Ortu (1994)), and the determination of aff $\Phi(S)$ is computationally easier than the determination of $\Phi^e(S)$, as it does not involve any non-negativity constraint. This is made clear by the following example.

EXAMPLE 4.1.   Consider an economy with two dates $(t = 0, 1)$, three possible states $(\Omega = \{\omega_1, \omega_2, \omega_3\})$, and two traded assets. The first asset is riskless and has dividend processes $d_1 = (0, 1, 1, 1)$. The second asset is risky, with dividend process $d_2 = (0, 1, 2, 3)$. Let $S_1 = (S_1(\xi_0), 0, 0, 0)$ and $S_2 = (S_2(\xi_0), 0, 0, 0)$ denote the price processes for the two securities. The set $\Phi^e(S)$ of extreme points of the closure of $\Phi(S)$ is then easily verified to be as follows:

$$\Phi^e(S) = \begin{cases} \left\{ \begin{pmatrix} 1 \\ 0 \\ 3S_1(\xi_0) - S_2(\xi_0) \\ S_2(\xi_0) - 2S_1(\xi_0) \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{3S_1(\xi_0) - S_2(\xi_0)}{2} \\ 0 \\ \frac{S_2(\xi_0) - S_1(\xi_0)}{2} \end{pmatrix} \right\}, & \begin{aligned} &\text{if } \tfrac{1}{3}S_2(\xi_0) \\ &< S_1(\xi_0) \\ &\leq \tfrac{1}{2}S_2(\xi_0); \end{aligned} \\[3em] \left\{ \begin{pmatrix} 1 \\ \frac{3S_1(\xi_0) - S_2(\xi_0)}{2} \\ 0 \\ \frac{S_2(\xi_0) - S_1(\xi_0)}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 2S_1(\xi_0) - S_2(\xi_0) \\ S_2(\xi_0) - S_1(\xi_0) \\ 0 \end{pmatrix} \right\}, & \begin{aligned} &\text{if } \tfrac{1}{2}S_2(\xi_0) \\ &< S_1(\xi_0) \\ &< S_2(\xi_0); \end{aligned} \\[3em] \emptyset, & \text{otherwise.} \end{cases}$$

The set $\Phi^e(S)$ is empty if $S_1(\xi_0) \leq \frac{1}{3}S_2(\xi_0)$ or $S_1(\xi_0) \geq S_2(\xi_0)$, since in this case there are arbitrage opportunities. The advantage of our characterization of feasible price processes over the one provided by He and Pearson (1991) is that when solving for an equilibrium the two cases $\frac{1}{3}S_2(\xi_0) < S_1(\xi_0) \leq \frac{1}{2}S_2(\xi_0)$ and $\frac{1}{2}S_2(\xi_0) < S_1(\xi_0) < S_2(\xi_0)$ need not be considered separately. In fact, either set of extreme points could be taken as our set $\hat{\Phi}(S)$, as they are both subsets of the affine hull of $\Phi(S)$ and span $X_S \perp$ (equivalently, they give rise to constraints in (14) that are a linear transformation of one another).

Using the second set of extreme points as the set $\hat{\Phi}(S)$ of Proposition 2, we then have that a consumption process $c \in \mathbb{R}^N_{++}$ is feasible for initial endowment e given the price system S if and only if it satisfies the two

linear constraints

$$(c(\xi_0) - e(\xi_0)) + \frac{3S_1(\xi_0) - S_2(\xi_0)}{2}(c(\xi_1) - e(\xi_1))$$

$$+\frac{S_2(\xi_0) - S_1(\xi_0)}{2}(c(\xi_3) - e(\xi_3)) = 0,$$

$$(c(\xi_0) - e(\xi_0)) + (2S_1(\xi_0) - S_2(\xi_0))(c(\xi_1) - e(\xi_1))$$

$$+(S_2(\xi_0) - S_1(\xi_0))(c(\xi_2) - e(\xi_2)) = 0,$$

where we have indexed the set of nodes by $(\xi_0, \xi_1, \xi_2, \xi_3)$. ∎

Since the characterization of feasible consumption processes in Proposition 2 contains no reference to the trading strategy θ that finances $c - e$, we can then reformulate the notion of equilibrium entirely in terms of the price process S and the consumption allocation $\{c_1^*, \ldots, c_I^*\}$.

DEFINITION 4.3.   An *effective equilibrium* for the economy $\mathcal{E}$ is a price process S and a set $c^* = \{c_i^* : i = 1, \ldots, I\}$ of consumption plans such that:

(i) c_i^* maximizes $U_i(c)$ on $B(e_i, S)$ for $i = 1, \ldots, I$,
(ii) the market for the consumption good clears: $\sum_{i=1}^{I} c_i^* = \bar{e}$.

The following proposition establishes the equivalence of equilibria and effective equilibria.

PROPOSITION 3.   *If (S, c^*) is an effective equilibrium for the economy $\mathcal{E}$, then there exists a set $\theta^* = (\theta_1^*, \ldots, \theta_I^*) \in \Theta^I$ of trading strategies such that (S, c^*, θ^*) is an equilibrium.*

Proof.   Since c_i^* is feasible for all i, there are trading strategies $\theta_i^* \in \Theta$ for $i = 1, \ldots, I - 1$ that finance the net trade $c_i^* - e_i$. Let $\theta_I^* = -\sum_{i=1}^{I-1} \theta_i^*$. Then the securities markets clear, and it is easy to check, using the clearing condition for the consumption good, that θ_I^* finances $c_I^* - e_I$. ∎

From now on we will find it convenient to use the notion of effective equilibria and will thus refer to a pair (S, c^*) as an equilibrium, without additional qualifications. For a given consumption policy $c \in \mathbb{R}_{++}^N$, let $\nabla^* U_i(c) = \nabla U_i(c)/\frac{\partial}{\partial c} u_i(c(\xi_0), \xi_0)$ denote the normalized gradient at c for agent i, i.e.,

$$\nabla^* U_i(c)(\xi) = \pi(\xi) \frac{\frac{\partial}{\partial c} u_i(c(\xi), \xi)}{\frac{\partial}{\partial c} u_i(c(\xi_0), \xi_0)}.$$

We then have the following characterization of an equilibrium.

PROPOSITION 4.   *A pair (S, c^*) is an equilibrium for the economy $\mathcal{E}$ if and only if the following conditions are satisfied for some $\hat{\Phi}(S) \subset \mathrm{aff}\,\Phi(S)$ such that $\mathrm{span}\hat{\Phi}(S) = X_S \perp$:*

$$\phi \cdot (c_i^* - e_i) = 0 \qquad \forall i, \; \forall \phi \in \hat{\Phi}(S) \tag{15}$$

$$\nabla^* U_i(c_i^*) \in \mathrm{span}\hat{\Phi}(S) \qquad \forall i \tag{16}$$

$$\sum_{i=1}^{I} c_i^* = \bar{e} \tag{17}$$

Proof.   First suppose that $\hat{\Phi}(S) \subset \mathrm{aff}\,\Phi(S)$ is such that $\mathrm{span}\hat{\Phi}(S) = X_S \perp$ and that (15)–(17) hold. From Proposition 2, the individual optimization problem can then be written as:

$$\max_{c_i \in \mathbb{R}_+^N} \; U_i(c_i) \tag{18}$$

$$\text{s.t.} \;\; \phi \cdot (c_i - e_i) = 0 \qquad \forall \phi \in \hat{\Phi}(S). \tag{19}$$

Under Assumptions 1–3, the non-negativity constraint is not binding and the Slater condition is satisfied, so that the necessary as well as sufficient conditions for this problem can be written as:

$$\nabla U_i(c_i^*) \; = \; \sum_{r=1}^{R} \psi^r \phi^r \tag{20}$$

$$\phi \cdot (c_i^* - e_i) \; = \; 0 \qquad\qquad \forall \phi \in \hat{\Phi}(S) \tag{21}$$

where $(\psi^1, \ldots, \psi^R)$ denotes the vector of Lagrange multipliers on the constraints in (19), and $r = 1, \ldots, R$ indexes the elements of $\hat{\Phi}(S)$. Clearly, (15)–(17) imply that (20)–(21) are satisfied for all i and that the market for the consumption good clears, so that (S, c^*) is an equilibrium. Conversely, suppose that (S, c^*) is an equilibrium and let $\hat{\Phi}(S)$ be any set of dim $X_S \perp$ linearly independent vectors from $\mathrm{aff}\,\Phi(S)$. Then Proposition 2 implies that c_i^* solves the program in (18)–(19), so that the first-order conditions in (20)–(21) are satisfied. Hence, (15)–(17) hold. ∎

In the above proposition, (15) expresses the feasibility constraint, and (17) the market clearing conditions. To interpret (16), observe that since $\nabla^* U_i(c_i^*) \gg 0$ and $\nabla^* U_i(c_i^*)(\xi_0) = 1$ for all i, $\nabla^* U_i(c_i^*) \in \mathrm{span}\hat{\Phi}(S)$ if and only if $\nabla^* U_i(c_i^*) \in \Phi(S)$. Therefore (16) requires that each agent's marginal rates of substitution constitute a valid set of state prices for S:

this is simply the first-order condition for the static problem of maximizing $U_i(c_i)$ over $B(e_i, S)$.

5. REPRESENTATIVE AGENT AND EQUILIBRIUM

Proposition 4 implies that in equilibrium the normalized gradient of each agent at his/her optimal consumption choice is a vector of state prices consistent with the equilibrium price S. This gives an expression for the equilibrium price process in terms of the optimal consumption policy c_i^* of any agent. It is also possible to relate equilibrium prices to the aggregate consumption process by introducing a representative agent.

DEFINITION 5.1.   A *representative agent* supporting the equilibrium (S, c^*) for the economy $\mathcal{E}$ is a utility function U such that $(S, \bar{e})$ is a (no-trade) equilibrium for the single-agent economy $\big((\Xi, \pi), U, \bar{e}, d\big)$.[12]

The existence of a representative agent that supports any competitive equilibrium in complete or essentially complete market economies is well known.[13] Moreover, if agents have homogeneous beliefs and time-additive, state-independent utilities, the representative agent also has time-additive and state-independent preferences and the same beliefs (see Constantinides (1982)). With incomplete markets, the construction of a representative agent is still possible, but his preferences are in general state-dependent, even when the individual agents have state-independent utilities and homogeneous beliefs. Before stating this result, we will need the following lemma:

LEMMA 1.   *Let* $\lambda = (\lambda_1, \ldots, \lambda_I) \in \mathbb{R}_{++}^{NI}$ *and define the function* $U(\cdot; \lambda) : \mathbb{R}_{++}^N \to \mathbb{R}$ *by:*

$$U(c; \lambda) = \sum_{\xi \in \Xi} \pi(\xi) u(c(\xi), \xi; \lambda) \tag{22}$$

where:

---

[12] A representative agent is in this setting simply a device to characterize equilibrium asset prices, and the representative agent's utility will in general depend not only on the individual utilities and beliefs, but also on the endowment distribution and the particular equilibrium (if there is more than one). The existence of a representative agent should therefore not be confused with the more demanding notion of *aggregation* of Gorman (1953) and Rubinstein (1974).

[13] Here, the term essentially complete refers to economies in which any conceivable new security is welfare-irrelevant. A typical case, often exploited in the financial literature, occurs when investors have homogeneous beliefs and time-additive, state-independent hyperbolic absolute risk aversion (HARA) utilities with the same cautioness parameter: the market is then essentially complete provided only that there exists a riskless asset in addition to the market portfolio.

$$u(c, \xi; \lambda) = \max_{z_1 + \cdots + z_I = c} \sum_{i=1}^{I} \lambda_i(\xi) u_i(z_i, \xi). \qquad (23)$$

Then $U(\cdot; \lambda)$ is an increasing and strictly concave additive utility function. Moreover, the functions $u(\cdot, \xi; \lambda)$ are continuously differentiable for all $\xi \in \Xi$ and satisfy the Inada conditions $\lim_{c \downarrow 0} \frac{\partial}{\partial c} u(c, \xi; \lambda) = \infty$ and $\lim_{c \uparrow \infty} \frac{\partial}{\partial c} u(c, \xi; \lambda) = 0$.

Proof. The first-order conditions for the maximization in (23) (which are necessary and sufficient for this problem given our assumptions) are:

$$\lambda_i(\xi) \frac{\partial}{\partial c} u_i(z_i^*, \xi) = \psi \qquad (i = 1, \ldots, I) \qquad (24)$$

$$\sum_{i=1}^{I} z_i^* = c \qquad (25)$$

where ψ denotes the Lagrange multiplier for the constraint in (25). Since $\frac{\partial}{\partial c} u_i(\cdot, \xi)$ is a strictly decreasing and continuously differentiable map from $\mathbb{R}_{++}$ onto itself, it has a strictly decreasing and continuously differentiable inverse function $f_i(\psi, \xi)$ mapping $\mathbb{R}_{++}$ onto itself. The function $K(\psi, \xi) = \sum_{i=1}^{I} f_i(\psi / \lambda_i(\xi), \xi)$ is therefore also a strictly decreasing and continuously differentiable map from $\mathbb{R}_{++}$ onto itself, and therefore it has a strictly decreasing and continuously differentiable inverse $\Psi(\cdot, \xi)$ mapping $\mathbb{R}_{++}$ onto itself. It then follows that

$$z_i^*(c, \xi; \lambda) = f_i\left(\frac{\Psi(c, \xi)}{\lambda_i(\xi)}, \xi \right). \qquad (26)$$

Since z_i^* is a continuously differentiable function of c, $u(\cdot, \xi; \lambda)$ is continuously differentiable. Moreover, we have:

$$
\begin{aligned}
\frac{\partial}{\partial c} u(c, \xi; \lambda) &= \sum_{i=1}^{I} \lambda_i(\xi) \frac{\partial}{\partial c} u_i\left[f_i\left(\frac{\Psi(c, \xi)}{\lambda_i(\xi)}, \xi \right), \xi \right] \qquad (27) \\
&= \Psi(c, \xi) \sum_{i=1}^{I} \frac{\partial}{\partial c} f_i\left(\frac{\Psi(c, \xi)}{\lambda_i(\xi)}, \xi \right) \\
&= \Psi(c, \xi) \frac{\partial}{\partial c} K(\Psi(c, \xi), \xi) = \Psi(c, \xi),
\end{aligned}
$$

so that U is strictly concave and the Inada conditions are satisfied. ∎

The following theorem establishes the existence of a representative agent with possibly incomplete financial markets.

THEOREM 1 (Existence of a representative agent).   *Define the utility function $U(c; \lambda)$ as in (22)–(23) and suppose that (S, c^*) is an equilibrium for the economy $\mathcal{E}$. Then $(S, \bar{e})$ is a (no trade) equilibrium for the representative agent with utility function $U(c; \lambda^*)$, where*

$$\lambda_i^*(\xi) = \frac{\frac{\partial}{\partial c} u_1(c_1^*(\xi), \xi)}{\frac{\partial}{\partial c} u_i(c_i^*(\xi), \xi)}. \tag{28}$$

The above utility function is increasing, strictly concave and continuously differentiable.   Moreover,   the   equilibrium   consumption   allocation $c^* = (c_1^*, \ldots, c_I^*)$ solves the problem (23) at $c = \bar{e}(\xi)$ for all $\xi \in \Xi$.

   Proof.   Let (S, c^*) be a given equilibrium for the economy $\mathcal{E}$. Define the representative agent's utility as in (22)–(23). We first show that when $c = \bar{e}(\xi)$, then the solution of the program in (23) is given by $\mathcal{C}_i(\xi; \lambda) \equiv z_i^*(\bar{e}(\xi), \xi; \lambda) = c_i^*(\xi)$. The first-order conditions in (24)–(25) give:

$$\frac{\frac{\partial}{\partial c} u_1(\mathcal{C}_1(\xi; \lambda), \xi)}{\frac{\partial}{\partial c} u_i(\mathcal{C}_i(\xi; \lambda), \xi)} = \lambda_i^*(\xi) = \frac{\frac{\partial}{\partial c} u_1(c_1^*(\xi), \xi)}{\frac{\partial}{\partial c} u_i(c_i^*(\xi), \xi)},$$

$$\sum_{i=1}^{I} \mathcal{C}_i(\xi; \lambda) = \bar{e}(\xi) = \sum_{i=1}^{I} c_i^*(\xi).$$

   Since the solution to (23) is unique, we have $\mathcal{C}_i(\xi; \lambda) = c_i^*(\xi)$ for all i. We now show that $(S, \bar{e})$ is an equilibrium for the economy with a single representative agent as above. Note that

$$\frac{\partial}{\partial c} u(\bar{e}(\xi), \xi; \lambda^*) = \sum_{i=1}^{I} \lambda_i^*(\xi) \frac{\partial}{\partial c} u_i(z_i^*(\bar{e}(\xi), \xi; \lambda^*), \xi) \frac{\partial}{\partial c} z_i^*(\bar{e}(\xi), \xi; \lambda^*)$$

$$= \frac{\partial}{\partial c} u_1(c_1^*(\xi), \xi) \left[\sum_{i=1}^{I} \frac{\partial}{\partial c} z_i^*(\bar{e}(\xi), \xi; \lambda^*) \right] = \frac{\partial}{\partial c} u_1(c_1^*(\xi), \xi).$$

   Therefore $\nabla^* U(\bar{e}; \lambda^*) = \nabla^* U_1(c_1^*)$, and hence, by Proposition 4, $\nabla^* U(\bar{e}; \lambda^*) \in \mathrm{span}\, \hat{\Phi}(S)$ for some $\hat{\Phi}(S) \subset \mathrm{aff}\, \Phi(S)$ such that $\mathrm{span}\, \hat{\Phi}(S) = X_S \perp$. By Proposition 4 again (with $I = 1$ and $U_1(\cdot) = U(\cdot; \lambda^*)$), this shows that $(S, \bar{e})$ is an equilibrium for the representative agent economy. The rest of the theorem follows from the previous lemma.   ∎

   Note that with incomplete markets the representative agent is not unique even within the class of utilities defined by (22)–(23). For example, it is immediate to see that if u_1 is replaced by u_j in (28), with $j \neq 1$, then

the weightings λ^* will in general change, but the resulting utility function $U(c; \lambda^*)$ still supports the equilibrium. Of course, in the special case in which the allocation is Pareto efficient (for which essentially complete markets suffice) the marginal rates of substitutions are equalized across agents, so that we have

$$\lambda_i^*(\xi) = \frac{\frac{\partial}{\partial c} u_1(c_1^*(\xi), \xi)}{\frac{\partial}{\partial c} u_i(c_i^*(\xi), \xi)} = \frac{\frac{\partial}{\partial c} u_1(c_1^*(\xi_0), \xi_0)}{\frac{\partial}{\partial c} u_i(c_i^*(\xi_0), \xi_0)} = \lambda_i^*(\xi_0),$$

for all i, and hence replacing u_1 with u_j in (28) amounts to simply multiplying the representative agent utility by a constant. It should also be clear from the above discussion that the Pareto efficiency of the equilibrium allocation is sufficient for the existence of a representative agent with state-independent utilities (constant weights).

Of course, the possibility of constructing a representative agent *given the equilibrium allocation* is hardly surprising. However, we now show that this construction proves fruitful *when searching for equilibria*.

Let the representative agent utility function $U(c; \lambda)$ be defined as in (22)–(23), and let $\Lambda = \{\iota\} \times \mathbb{R}_{++}^{N(I-1)}$, where $\iota = (1, \dots, 1) \in \mathbb{R}^N$. Theorem 1 (together with equations (26)–(27)) implies that if (S, c^*) is an equilibrium for the economy $\mathcal{E}$, then there exists a vector $\lambda^* = (\iota, \lambda_{-1}^*) \in \Lambda$ such that, for all $\xi \in \Xi$:

$$S(\xi) = \mathcal{S}(\xi; \lambda^*) = \sum_{\hat{\xi} \succ \xi} \frac{\nabla^* U(\bar{e}; \lambda^*)(\hat{\xi})}{\nabla^* U(\bar{e}; \lambda^*)(\xi)} d(\hat{\xi}) = \sum_{\hat{\xi} \succ \xi} \frac{\pi(\hat{\xi})}{\pi(\xi)} \frac{u_c(\bar{e}(\hat{\xi}), \hat{\xi}; \lambda^*)}{u_c(\bar{e}(\xi), \xi; \lambda^*)} d(\hat{\xi})$$

$$(29)$$

and

$$c_i^*(\xi) = \mathcal{C}_i(\xi; \lambda^*) = f_i\left(\frac{u_c(\bar{e}(\xi), \xi; \lambda^*)}{\lambda_i^*(\xi)}, \xi\right). \tag{30}$$

This shows that both the equilibrium price process and the equilibrium allocation can be expressed as a function of the vector λ^* of weights in the representative agent utility and that it should therefore be possible to reformulate the conditions for an equilibrium entirely in terms of conditions on λ^*. This program has been carried out by Negishi (1960) in the context of Arrow-Debreu economies and by Karatzas, Lehoczky and Shreve (1990) and Karatzas, Lakner, Lehoczky and Shreve (1991) in the context of financial economies with complete securities markets and continuous trading.

The following theorem establishes a similar characterization for our incomplete-market economy.

THEOREM 2. *A security price process S and an allocation $c^* = (c_1^*, \dots, c_I^*)$ constitute an equilibrium for the economy $\mathcal{E}$ if and only if S and c^* are as in*

(29)–(30) for some $\lambda^* \in \Lambda$ and there exists a set $\hat{\Phi}(S) \subset \text{aff}\,\Phi(S)$ such that $\text{span}\hat{\Phi}(S) = X_S \perp$ and the following two conditions hold for $i = 2, \ldots, I$:

$$\nabla^* U_i(c_i^*) \in \text{span}\hat{\Phi}(S) \tag{31}$$

$$\phi \cdot (c_i^* - e_i) = 0 \qquad \forall \phi \in \hat{\Phi}(S) \tag{32}$$

Proof. First, suppose that (29)–(30) and (31)–(32) are satisfied for $i = 2, \ldots, I$. Then (31) guarantees that (16) is satisfied for $i = 2, \ldots, I$. That this condition is satisfied for $i = 1$ follows from (29) and the equality $\nabla^* U_1(c_1^*) = \nabla^* U_1(\mathcal{C}_1(\lambda^*)) = \nabla^* U(\bar{e}; \lambda^*)$. Next, condition (32) guarantees that $c_i^* \in B(e_i, S)$ for $i = 2, \ldots, I$. That $c_1^* \in B(e_i, S)$ follows from $\sum_{i=1}^I c_i^* = \sum_{i=1}^I \mathcal{C}_i(\lambda^*) = \bar{e}$. Therefore, the conditions (15)–(17) for an equilibrium are satisfied.

On the other end, suppose that the process (S, c^*) is an equilibrium. Then conditions (15)–(17) are satisfied and it should be clear from the proof of Theorem 1 that this implies that (29)–(30) and (31)–(32) are satisfied for λ^* defined as in (28). ▌

Note that since $S = \mathcal{S}(\lambda^*)$ and $c_i^* = \mathcal{C}_i(\lambda^*)$, equations (31) and (32) are in fact restrictions on the vector of equilibrium weightings λ^*. Condition (31) places a limit on the variability of each λ_i^* across time and states. In particular, if markets are complete, then $\Phi(\mathcal{S}(\lambda^*))$ consists of a unique element, $\nabla^* U(\bar{e}; \lambda^*)$. Since

$$\nabla^* U_i(\mathcal{C}_i(\lambda^*))(\xi) = \frac{\lambda_i^*(\xi_0)}{\lambda_i^*(\xi)} \nabla^* U(\bar{e}; \lambda^*)(\xi),$$

we have

$$\nabla^* U_i(\mathcal{C}_i(\lambda^*)) \in \text{span}\hat{\Phi}(\mathcal{S}(\lambda^*)) \iff \frac{\lambda_i^*(\xi_0)}{\lambda_i^*(\xi)} \nabla^* U(\bar{e}; \lambda^*)(\xi) = \nabla^* U(\bar{e}; \lambda^*)(\xi) \; \forall \xi \in \Xi$$

so that the above condition restricts λ_i^* to be constant for each individual i. Condition (32) is a generalization of the condition provided by Negishi (1960) and Karatzas *et alii* (1990, 1991) for complete markets: in that case, condition (32) collapses into a single budget constraint with respect to the unique Arrow-Debreu price system $\nabla^* U(\bar{e}; \lambda^*)$.

6. COMPUTATION OF EQUILIBRIA

The conditions derived in the previous section can be effectively used when searching numerically for equilibria. Let $\hat{\Phi}(S)$ be a set of R linearly independent solutions of the linear system $\phi^\top X_S = 0$. Evaluating the

above solutions at $S = \mathcal{S}(\lambda)$ gives the set $\hat{\Phi}(\mathcal{S}(\lambda)) = \{\phi^1(\lambda), \dots, \phi^R(\lambda)\}$. Introducing a vector $\alpha \in \mathbb{R}^{R(I-1)}$ of auxiliary variables and using the fact that

$$\nabla^* U_i(\mathcal{C}_i(\lambda))(\xi) = \pi(\xi) \frac{u_c(\bar{e}(\xi), \xi; \lambda)/\lambda_i(\xi)}{u_c(\bar{e}(\xi_0), \xi_0; \lambda)/\lambda_i(\xi_0)},$$

conditions (31)–(32) can be rewritten as the following non-linear system of $(N + R)(I - 1)$ equations in the $(N + R)(I - 1)$-dimensional vector of unknowns (λ_{-1}, α):

$$\begin{cases} 0 = \displaystyle\sum_{r=1}^R \alpha_i^r \phi^r(\xi; \lambda) - \pi(\xi) \frac{u_c(\bar{e}(\xi), \xi; \lambda)/\lambda_i(\xi)}{u_c(\bar{e}(\xi_0), \xi_0; \lambda)/\lambda_i(\xi_0)} & (i = 2, \dots, I; \ \xi \in \Xi) \\ 0 = \displaystyle\sum_{\xi \in \Xi} \phi^r(\xi; \lambda) \left[f_i \left(\frac{u_c(\bar{e}(\xi), \xi; \lambda)}{\lambda_i(\xi)}, \xi \right) - e_i(\xi) \right] & (i = 2, \dots, I; \ r = 1, \dots, R). \end{cases}$$

(33)

Theorem 2 implies that if $(\lambda_{-1}^*, \alpha^*)$ solves the above system and $\text{span}\hat{\Phi}(\mathcal{S}(\lambda^*)) = X_{\mathcal{S}(\lambda^*)} \perp$, then $(\mathcal{S}(\lambda^*), \{\mathcal{C}_1(\lambda^*), \dots, \mathcal{C}_I(\lambda^*)\})$ is an equilibrium for the economy $\mathcal{E}$. The optimal trading strategies can also be determined by solving the linear systems $X_{\mathcal{S}(\lambda^*)} \theta_i^* = \mathcal{C}_i(\lambda^*) - e_i$ for θ_i^*. The main advantage of this procedure over the traditional numerical approach based on stochastic dynamic programming is that it does not require the determination of the agents' demand functions and hence the solution of the KI Bellman equations.[14] In other words, there is no need to determine the optimal policies at each node for all possible levels of wealth and all possible no-arbitrage prices.

In order to write down the above system of equations, the only preliminary step required is the determination of a set $\hat{\Phi}(S)$. In this regard, it is important to note that when $T \geq 2$ the dimension of $X_S \perp$ is not known a priori, as it depends on the specific equilibrium price process $S = \mathcal{S}(\lambda^*)$. However, it is still possible to determine (at least generically) the number R of linearly independent pseudo state prices needed to characterize the set of feasible consumption processes. First, note that since the space spanned by the columns of X_S depends on the investment opportunities available at each non-terminal node of the tree, the column rank of X_S can be characterized as follows (see Magill and Quinzii (1996) for a proof). For $\xi \in \Xi^-$, let $S(\xi^+)$ and $d(\xi^+)$ denote the $K \times b(\xi)$ matrices with generic elements $S_k(\hat{\xi})$ and $d_k(\hat{\xi})$, respectively, with $\hat{\xi} \in \xi^+$. Then, provided that S admits no arbitrage opportunities,

$$\text{rank} X_S = \sum_{\xi \in \Xi^-} \text{rank}(S(\xi^+) + d(\xi^+)) \leq \sum_{\xi \in \Xi^-} \min(b(\xi), K).$$

---

[14]See Kehoe (1991) and Judd (1998) for an overview of numerical approaches to the computation of rational expectations equilibria.

As an intermediate step in proving existence of equilibria, Duffie and Schafer (1985, 1986) have shown that the above inequality will indeed hold as an equality for the generic set A of economies in Proposition 1. Therefore, for this generic set of economies,

$$\dim X_S \perp = N - \mathrm{rank} X_S = 1 + \sum_{\xi \in \Xi^-} \max[0, b(\xi) - K], \qquad (34)$$

so that choosing $R = 1 + \sum_{\xi \in \Xi^-} \max[0, b(\xi) - K]$ linearly independent pseudo state prices will be sufficient for all but an exceptional set of economies.

It is also worth noting that since $\phi^\top X_S = 0$ is satisfied if and only if

$$S(\xi) = \sum_{\hat{\xi} \in \xi^+} \frac{\phi(\hat{\xi})}{\phi(\xi)} (S(\hat{\xi}) + d(\hat{\xi}))$$

holds for all $\xi \in \Xi^-$, it is possible to generate the pseudo state prices by solving a series of simpler problems, as follows. For each non-terminal node $\xi \in \Xi^-$, define a set of conditional pseudo state prices for the immediate successors of ξ to be a vector $\phi_\xi \in \mathbb{R}^{b(\xi)}$ solving

$$S(\xi) = [S(\xi^+) + d(\xi^+)]\phi_\xi. \qquad (35)$$

Each (unconditional) pseudo state price vector ϕ can then be obtained by choosing a set of conditional pseudo state prices for each non-terminal node and letting $\phi(\xi)$ be 1 if $\xi = \xi_0$ and otherwise be the product of the chosen conditional state prices along the path leading from the ξ_0 to ξ. If the computation of equilibria is repeated for different specifications of preferences and endowments, then the construction of the set of pseudo state prices has to be performed only once. The above discussion can be clarified by considering some examples.

EXAMPLE 6.1.   Let us consider again the two-date, two-asset economy presented in example 1, i.e., $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $d^1 = (0, 1, 1, 1)$ and $d^2 = (0, 1, 2, 3)$. Write $\Xi = (\xi_0, (\xi_1, \xi_2, \xi_3))$ and suppose that the economy is populated by two agents with beliefs represented by the vector of node probabilities $\pi = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ and utilities

$$U_1(c_1) = \sum_{\xi \in \Xi} \pi(\xi) \rho_1(\xi) \log c_1(\xi)$$

and

$$U_2(c_2) = -\sum_{\xi \in \Xi} \pi(\xi) \rho_2(\xi) c_2(\xi)^{-1},$$

respectively, with $\rho_1 = \rho_2 = (1, .9, .9, .9)$. Also, assume that $e_1 = (1, 1, 1, 1)$ and $e_2 = (1, 1, 2, 3)$, so that $\bar{e} = e_1 + e_2 = (2, 2, 3, 4)$. We demonstrate below how the equilibrium prices $S_1(\xi_0)$ and $S_2(\xi_0)$ can be determined using the "representative agent" approach.

Let $\lambda = (\iota, \lambda_2)$ be given. The program in (23) can be solved analytically to give:[15]

$$u_c(\bar{e}(\xi), \xi; \lambda) = \frac{2\rho_1(\xi)\bar{e}(\xi) + \rho_2(\xi)\lambda_2(\xi) + \sqrt{\rho_2(\xi)\lambda_2(\xi)(4\rho_1(\xi)\bar{e}(\xi) + \rho_2(\xi)\lambda_2(\xi))}}{2\bar{e}(\xi)^2},$$

$$\mathcal{C}_1(\xi; \lambda) = \rho_1(\xi)/u_c(\bar{e}(\xi); \lambda),$$

and

$$\mathcal{C}_2(\xi; \lambda) = \sqrt{(\rho_2(\xi)\lambda(\xi))/u_c(\bar{e}(\xi); \lambda)}.$$

The time-0 security prices are given in terms of λ by

$$\mathcal{S}_1(\xi_0; \lambda) = \frac{1}{2}\frac{u_c(2, \xi_1; \lambda)}{u_c(2, \xi_0; \lambda)} \times 1 + \frac{1}{3}\frac{u_c(3, \xi_2; \lambda)}{u_c(2, \xi_0; \lambda)} \times 1 + \frac{1}{6}\frac{u_c(4, \xi_3; \lambda)}{u_c(2, \xi_0; \lambda)} \times 1$$

and

$$\mathcal{S}_2(\xi_0; \lambda) = \frac{1}{2}\frac{u_c(2, \xi_1; \lambda)}{u_c(2, \xi_0; \lambda)} \times 1 + \frac{1}{3}\frac{u_c(3, \xi_2; \lambda)}{u_c(2, \xi_0; \lambda)} \times 2 + \frac{1}{6}\frac{u_c(4, \xi_3; \lambda)}{u_c(2, \xi_0; \lambda)} \times 3.$$

From Example 1, we know that we can take

$$\hat{\Phi}(S) = \left\{ \begin{pmatrix} 1 \\ \frac{3S_1(\xi_0) - S_2(\xi_0)}{2} \\ 0 \\ \frac{S_2(\xi_0) - S_1(\xi_0)}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 2S_1(\xi_0) - S_2(\xi_0) \\ S_2(\xi_0) - S_1(\xi_0) \\ 0 \end{pmatrix} \right\}.$$

---

[15]For other preferences specifications, the computation of u_c will in general have to be performed numerically.

Letting $\alpha = (\alpha_2^1, \alpha_2^2)$, we can then write the system in (33) as

$$
\begin{cases}
0 = \alpha_2^1 + \alpha_2^2 - 1 \\
0 = \alpha_2^1 \dfrac{3\mathcal{S}_1(\xi_0; \lambda) - \mathcal{S}_2(\xi_0; \lambda)}{2} + \alpha_2^2 (2\mathcal{S}_1(\xi_0; \lambda) - \mathcal{S}_2(\xi_0; \lambda)) \\
\quad - \dfrac{1}{2} \dfrac{u_c(2, \xi_1; \lambda)/\lambda_2(\xi_1)}{u_c(2, \xi_0; \lambda)/\lambda_2(\xi_0)} \\
0 = \alpha_2^2 (\mathcal{S}_2(\xi_0; \lambda) - \mathcal{S}_1(\xi_0; \lambda)) - \dfrac{1}{3} \dfrac{u_c(3, \xi_2; \lambda)/\lambda_2(\xi_2)}{u_c(2, \xi_0; \lambda)/\lambda_2(\xi_0)} \\
0 = \alpha_2^1 \dfrac{\mathcal{S}_2(\xi_0; \lambda) - \mathcal{S}_1(\xi_0; \lambda)}{2} - \dfrac{1}{6} \dfrac{u_c(4, \xi_3; \lambda)/\lambda_2(\xi_3)}{u_c(2, \xi_0; \lambda)/\lambda_2(\xi_0)} \\
0 = (\mathcal{C}_2(\xi_0; \lambda) - 1) + \dfrac{3\mathcal{S}_1(\xi_0; \lambda) - \mathcal{S}_2(\xi_0; \lambda)}{2} (\mathcal{C}_2(\xi_1; \lambda) - 1) \\
\quad + \dfrac{\mathcal{S}_2(\xi_0; \lambda) - \mathcal{S}_1(\xi_0; \lambda)}{2} (\mathcal{C}_2(\xi_3; \lambda) - 3) \\
0 = (\mathcal{C}_2(\xi_0; \lambda) - 1) + (2\mathcal{S}_1(\xi_0; \lambda) - \mathcal{S}_2(\xi_0; \lambda))(\mathcal{C}_2(\xi_1; \lambda) - 1) \\
\quad + (\mathcal{S}_2(\xi_0; \lambda) - \mathcal{S}_1(\xi_0; \lambda))(\mathcal{C}_2(\xi_2; \lambda) - 2)
\end{cases}
$$

The first four equations impose the condition that the shadow prices assigned by the two agents to the traded securities coincide. The other two equations represent the feasibility constraints on the consumption pattern of the second agent.

The vector $(\lambda_2^*, \alpha^*) = (1.402, 1.409, 1.367, 1.454, 0.397, 0.603)$ solves the above system. The corresponding equilibrium stock prices are $S_1(\xi_0) = 0.683$ and $S_2(\xi_0) = 0.975$, while the optimal consumption policies are

$$c_1^* = (0.886, 0.884, 1.546, 2.208) \qquad \text{and} \qquad c_2^* = (1.114, 1.116, 1.454, 1.792).$$

The investment strategies that finance the chosen consumption policies are

$$\theta_1^* = (-0.778, 0.662) \qquad \text{and} \qquad \theta_2^* = (0.778, -0.662). \quad \blacksquare$$

To summarize, we have shown that by using constructing a representative agent we can reduce the problem of searching for a dynamic equilibrium (via dynamic programming method) to a problem of finding a solution to a system of non-linear algebraic equations where the non-linear functions involved can be derived explicitly and conveniently from the utility functions of individual investors. Specifically, whenever there is an analytical solution for the utility function of the representative agent U, these non-linear functions also have an analytical expression. If there is no analytical expression for U, then numerical calculation of U as a function of λ evaluated at the aggregate endowment would still lead to a well-defined system

of non-linear algebraic equations as specified in (33). Standard numerical techniques such as the Newton-Raphson and successive over-relaxation methods can now be applied to solve the system of non-linear equations. We refer readers to Ortega and Rheinboldt (1070) and Press, etc. (1992) for more details.

It is useful to note that the success of our representative agent approach is parallel to the success we have already seen in solving individuals' consumption and investment problems using the martingale representation technique. The main idea there is to map an individual's dynamic consumption and investment problem into a static one so that it can be handled with relative ease. The idea of the representative agent approach developed here is similar in the sense that we are transforming a dynamic general equilibrium problem into a static one in which the search for equilibrium is reduced to the search for the λ weightings at different dates, states and for different agents, while the search for the λ weightings involves solving a system of non-linear equations. In addition, searching over the weightings λ, rather than over the portfolio policies θ_i of the individual agents—as in the traditional approach based on the stochastic Euler equations of dynamic programming, results in a much more compact vector of unknowns when the number of traded securities is large.

The computation of a set $\hat{\Phi}(S)$ of pseudo state prices points is only slightly more complicated when more than two periods are involved. The following example illustrates how this can be done.

EXAMPLE 6.2.   Consider an economy with three dates $(t = 0, 1, 2)$. Assume that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_8\}$ and that the information structure is given by $F_0 = \Omega$,

$$F_1 = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8\}\},$$

and $F_2 = \{\{\omega_1\}, \{\omega_2\}, \ldots, \{\omega_8\}\}$. There are two securities: one is a riskless bond with dividend process $d_1 = (0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, and the other is a stock with dividend process $d_2 = (0, 1, 2, 3, 1, 2, 3, 4, 5, 6, 7, 8)$. Let $\Xi = \{\xi_0, \xi_1, \ldots, \xi_{11}\}$ denote the set of nodes and $\Xi^- = \{\xi_0, \xi_1, \xi_2, \xi_3\}$ be the set of non-terminal nodes.

For any $\xi \in \Xi^-$, the conditional pseudo state price vectors ϕ_ξ solve:

$$\begin{cases} S_1(\xi) = \sum_{\hat{\xi} \in \xi^+} \phi_\xi(\hat{\xi})[S_1(\hat{\xi}) + d_1(\hat{\xi})] \\ S_2(\xi) = \sum_{\hat{\xi} \in \xi^+} \phi_\xi(\hat{\xi})[S_2(\hat{\xi}) + d_2(\hat{\xi})] \end{cases} \tag{36}$$

Assuming that $\frac{S_2(\xi_0)}{S_1(\xi_0)} \neq \frac{S_2(\xi_1)+1}{S_1(\xi_1)+1}$, any solution of the above system for $\xi \in \Xi^- \backslash \{\xi_3\}$ can be written as a linear combination of the following two

conditional pseudo state prices

$$
\hat{\phi}_\xi^1 = \begin{pmatrix} \frac{S_1(\xi)[S_2(\xi_{+3})+d_2(\xi_{+3})]-S_2(\xi)[S_1(\xi_{+3})+d_1(\xi_{+3})]}{[S_1(\xi_{+1})+d_1(\xi_{+1})][S_2(\xi_{+3})+d_2(\xi_{+3})]-[S_1(\xi_{+3})+d_1(\xi_{+3})][S_2(\xi_{+1})+d_2(\xi_{+1})]} \\ 0 \\ \frac{S_2(\xi)[S_1(\xi_{+1})+d_1(\xi_{+1})]-S_1(\xi)[S_2(\xi_{+1})+d_2(\xi_{+1})]}{[S_1(\xi_{+1})+d_1(\xi_{+1})][S_2(\xi_{+3})+d_2(\xi_{+3})]-[S_1(\xi_{+3})+d_1(\xi_{+3})][S_2(\xi_{+1})+d_2(\xi_{+1})]} \end{pmatrix},
$$

and

$$
\hat{\phi}_\xi^2 = \begin{pmatrix} \frac{S_1(\xi)[S_2(\xi_{+2})+d_2(\xi_{+2})]-S_2(\xi)[S_1(\xi_{+2})+d_1(\xi_{+2})]}{[S_1(\xi_{+1})+d_1(\xi_{+1})][S_2(\xi_{+2})+d_2(\xi_{+2})]-[S_1(\xi_{+2})+d_1(\xi_{+2})][S_2(\xi_{+1})+d_2(\xi_{+1})]} \\ \frac{S_2(\xi)[S_1(\xi_{+1})+d_1(\xi_{+1})]-S_1(\xi)[S_2(\xi_{+1})+d_2(\xi_{+1})]}{[S_1(\xi_{+1})+d_1(\xi_{+1})][S_2(\xi_{+2})+d_2(\xi_{+2})]-[S_1(\xi_{+2})+d_1(\xi_{+2})][S_2(\xi_{+1})+d_2(\xi_{+1})]} \\ 0 \end{pmatrix},
$$

where we have indexed the elements in ξ_+ by $(\xi_{+1}, \xi_{+2}, \xi_{+3})$. When $\xi = \xi_3$, the system in (36) has a unique solution given by

$$
\hat{\phi}_{\xi_3} = \begin{pmatrix} 8S_1(\xi_3) - S_2(\xi_3) \\ S_2(\xi_3) - 7S_1(\xi_3) \end{pmatrix}.
$$

The above conditional state prices lead to $2 \times 2 \times 2 \times 1 = 8$ possible choices of unconditional state prices. After discarding linearly dependent vectors, we are led to the following set $\hat{\Phi}(S)$, shown in partitioned form:

$$
\hat{\Phi}(S) = \left\{ \begin{pmatrix} 1 \\ \cdots \\ \hat{\phi}_{\xi_0}^1 \\ \cdots \\ \hat{\phi}_{\xi_0}^1(\xi_1)\,\hat{\phi}_{\xi_1}^1 \\ \cdots \\ 0 \\ \cdots \\ \hat{\phi}_{\xi_0}^1(\xi_3)\,\hat{\phi}_{\xi_3}^1 \end{pmatrix}, \begin{pmatrix} 1 \\ \cdots \\ \hat{\phi}_{\xi_0}^1 \\ \cdots \\ \hat{\phi}_{\xi_0}^1(\xi_1)\,\hat{\phi}_{\xi_1}^2 \\ \cdots \\ 0 \\ \cdots \\ \hat{\phi}_{\xi_0}^1(\xi_3)\,\hat{\phi}_{\xi_3}^1 \end{pmatrix}, \begin{pmatrix} 1 \\ \cdots \\ \hat{\phi}_{\xi_0}^2 \\ \cdots \\ \hat{\phi}_{\xi_0}^2(\xi_1)\,\hat{\phi}_{\xi_1}^1 \\ \cdots \\ \hat{\phi}_{\xi_0}^2(\xi_2)\,\hat{\phi}_{\xi_2}^1 \\ \cdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \cdots \\ \hat{\phi}_{\xi_0}^2 \\ \cdots \\ \hat{\phi}_{\xi_0}^2(\xi_1)\,\hat{\phi}_{\xi_1}^1 \\ \cdots \\ \hat{\phi}_{\xi_0}^2(\xi_2)\,\hat{\phi}_{\xi_2}^2 \\ \cdots \\ 0 \end{pmatrix} \right\},
$$

where 0 denotes the zero vector in either $\mathbb{R}^3$ or $\mathbb{R}^2$. Note that in this case

$$
1 + \sum_{\xi \in \Xi^-} \max[0, b(\xi) - K] = 4,
$$

so that 4 linearly independent pseudo state prices are generically sufficient to span the orthogonal space of $X_{\mathcal{S}(\lambda^*)}$.

Assuming the same preference specification as in the previous example, with

$$
\rho_1 = \rho_2 = (1, .9, .9, .9, .81, .81, .81, .81, .81, .81, .81, .81),
$$

$$\pi = \left(1, \frac{3}{8}, \frac{3}{8}, \frac{2}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right),$$

and endowment processes

$$e_1 = (1, 1, 1, 1, 1, 2, 3, 4, 1, 2, 3, 4)$$

$$e_2 = (1, 1, 2, 3, 1, 1, 2, 2, 3, 3, 4, 4),$$

the vector $(\lambda^*_{-1}, \alpha^*)$ with

$$\lambda^*_{-1} = (1.660, 1.686, 1.549, 1.835, 1.894, 1.157, 2.583, 0.921, 3.426, 1.008, 1.835, 1.835)$$

and $\alpha^* = (-0.122, 0.509, 0.476, 0.137)$ solves the system (33). Since for this choice of λ^* the condition $\mathrm{span}\hat{\Phi}(\mathcal{S}(\lambda^*)) = X_{\mathcal{S}(\lambda^*)} \perp$ is satisfied, we conclude that λ^* defines an equilibrium. The corresponding equilibrium prices and consumption and investment policies are given by

$$S_1 = (0.946, 0.567, 0.521, 0.405, 0, 0, 0, 0, 0, 0, 0, 0),$$

$$S_2 = (2.070, 0.918, 2.634, 3.024, 0, 0, 0, 0, 0, 0, 0, 0),$$

$$c^*_1 = (0.828, 0.822, 1.484, 2.057, 0.783, 1.628, 2.473, 4.065, 1.634, 3.203, 4.218, 4.978),$$

$$c^*_2 = (1.172, 1.178, 1.516, 1.943, 1.217, 1.372, 2.527, 1.935, 2.366, 1.797, 2.782, 3.022),$$

$$\theta^*_1 = (-0.746, 0.424, -0.063, -0.155, -2.212, 0.569, 2.899, -0.240)$$

and $\theta^*_2 = -\theta^*_1$.   ▌

7. ONE-PERIOD SECURITIES

The determination of a spanning set $\hat{\Phi}(S)$ and the system in (33) are simplified in the case in which the market consists of a sequence of one-period securities.[16] Let X_S denote the $N \times K$ matrix describing the opportunity set, where K now represents the total number of one-period securities available in the economy. Let $D = X_0$ denote the $N \times K$ dividend matrix and A the $N \times N$ matrix with $A_{ij} = 1$ if $i = j$ or $\xi_j \succ \xi_i$, and $A_{ij} = 0$

---

[16] One-period securities are securities that pay a liquidating dividend one period after they are created. Thus, new one-period securities must be created over time for each node.

otherwise. Then A has full rank and it is easy to see that, if $\phi \in \Phi(S)$ is any set of state prices, we have

$$A\mathrm{diag}(\phi)X_S = \mathrm{diag}(\phi)D$$

and hence

$$X_S = \mathrm{diag}(\phi)^{-1}A^{-1}\mathrm{diag}(\phi)D.$$

This implies that if K spans $D \perp$, then $\mathrm{diag}(\phi)A^\top\mathrm{diag}(\phi)^{-1}K$ spans $X_S \perp$, and we have the following result, where we use the notation

$$\mathrm{vec}\left(g(\bar{e}, \lambda)\right) = \left(g(\bar{e}(\xi_1), \lambda(\xi_1)), \ldots, g(\bar{e}(\xi_N), \lambda(\xi_N))\right)^\top.$$

PROPOSITION 5. *Suppose that securities are short-lived. A security price process S and an allocation $c^* = (c_1^*, \ldots, c_I^*)$ constitute an equilibrium for the economy $\mathcal{E}$ if and only if S and c^* are as in (29)–(30) for some $\lambda^* \in \Lambda$ solving*

$$\begin{cases} 0 = D^\top diag(\nabla^*U(\bar{e};\lambda))(A^\top)^{-1}vec(\lambda_i^{-1}) & (i = 2, \ldots, I) \\ 0 = K^\top diag(\nabla^*U(\bar{e};\lambda))^{-1}Adiag(\nabla^*U(\bar{e};\lambda))vec\left(f_i\left(\frac{u_c(\bar{e};\lambda)}{\lambda_i}\right) - e_i\right) & (i = 2, \ldots, I) \end{cases}$$
$$(37)$$

where D denotes the dividend matrix and K is any matrix spanning $D \perp$. Moreover, a solution to the above system exists.

Proof. By theorem 2, a security price process S and an allocation $(c_1^*, \cdots, c_I^*)$ constitute an equilibrium if and only if $S = \S(\lambda^*)$ and $c_i^* = \mathcal{C}(\lambda^*)$ for some $\lambda^* \in \Lambda$ such that $\nabla^*U_i(\mathcal{C}_i(\lambda^*)) \in X_{\S(\lambda^*)} \perp$ and $\mathcal{C}_i(\lambda^*) - e_i \in \mathrm{span}X_{\S(\lambda^*)}$ for all i. Since

$$X_{\S(\lambda^*)} = \mathrm{diag}(\nabla^*U(\bar{e};\lambda^*))^{-1}A^{-1}\mathrm{diag}(\nabla^*U(\bar{e};\lambda^*))D$$

and $\nabla^*U_i(\mathcal{C}_i(\lambda^*))$ is proportional to $\mathrm{diag}(\nabla^*U(\bar{e};\lambda^*))\mathrm{vec}(\lambda_i^{-1})$, this implies that an equilibrium exists if and only if there exists a solution to (37). ∎

The existence of an equilibrium with short-lived assets, and hence of a solution to (37), is a well known result. For completeness, a direct proof of the existence of a solution to (37) using degree theory is provided in the Appendix.[17]

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[17]While we conjecture that Proposition 1 could potentially be proved using our representative agent approach, we didn't pursue this task in this paper. Hopefully, the proof of existence for this simplified case can serve as an illustration.

8. CONCLUSION

This paper has shown that a rational expectations equilibrium in an economy with discrete trading and a finite number of states can be sustained by a representative agent, provided only that the agents' preferences are additively separable and that some mild technical conditions hold. A Pareto efficient allocation (essentially complete markets) corresponds to a representative agent with state-independent preferences. Using the previous result on the existence of a representative agent, theorem 2 reduces the search for an equilibrium to the solution of a system of nonlinear equations in the unknown components of the vector λ that characterizes the representative agent's preferences. This representation of an equilibrium can be usefully exploited for numerical computation.

APPENDIX

This Appendix provides a direct proof of the existence of a solution to the system in (37). For ease of notation, we consider only the case $I = 2$ (the general case is similar).

Define the maps $F : \Lambda \to \mathbb{R}^N$, $G : \Lambda \to \mathbb{R}^N$ and $H : \Lambda \times [0,1] \to \mathbb{R}^N$ by

$$
F(\lambda) = \begin{pmatrix} D^\top \mathrm{diag}(\nabla^* U(\bar{e};\lambda))(A^\top)^{-1}\mathrm{vec}(\lambda_2^{-1}) \\ K^\top \mathrm{diag}(\nabla^* U(\bar{e};\lambda))^{-1} A\mathrm{diag}(\nabla^* U(\bar{e};\lambda))\mathrm{vec}\left(f_2\left(\frac{u_c(\bar{e};\lambda)}{\lambda_2}\right) - e_2\right) \end{pmatrix},
$$

$$
G(\lambda) = \begin{pmatrix} D^\top \mathrm{diag}(\nabla^* U(\bar{e};\lambda))(A^\top)^{-1} \\ K^\top \mathrm{diag}(\nabla^* U(\bar{e};\lambda))^{-1} A\mathrm{diag}(\nabla^* U(\bar{e};\lambda)) \end{pmatrix} \mathrm{vec}(\lambda_2^{-1} - 1),
$$

and $H(\lambda, t) = (1-t)G(\lambda) + tF(\lambda)$. Clearly, (37) is equivalent to $F(\lambda) = 0$. Also, since the $N \times N$ matrix

$$
\begin{pmatrix} D^\top \mathrm{diag}(\nabla^* U(\bar{e};\lambda))(A^\top)^{-1} \\ K^\top \mathrm{diag}(\nabla^* U(\bar{e};\lambda))^{-1} A\mathrm{diag}(\nabla^* U(\bar{e};\lambda)) \end{pmatrix}
$$

has full rank for all $\lambda \in \Lambda$, the equation $G(\lambda) = 0$ has a unique solution $\lambda = \iota$. By the degree theorem in the Appendix of Geanakoplos and Shafer (1990), in order to show that the equation $F(\lambda) = 0$, and hence (37), have a solution, we only need to show that $H^{-1}(0)$ is compact (and hence so are $F^{-1}(0)$ and $G^{-1}(0)$).

To this end, we start by observing that $(\lambda, t) \in H^{-1}(0)$ implies

$$
\begin{aligned}
0 &= D^\top \mathrm{diag}(\nabla^* U(\bar{e};\lambda))(A^\top)^{-1}\mathrm{vec}\left(\lambda_2^{-1} - (1-t)\right) \\
0 &= K^\top \mathrm{diag}(\nabla^* U(\bar{e};\lambda))^{-1} A\mathrm{diag}(\nabla^* U(\bar{e};\lambda)) \\
&\quad \times \mathrm{vec}\left((1-t)(\lambda_2^{-1} - 1) + t\left(f_2\left(\frac{u_c(\bar{e};\lambda)}{\lambda_2}\right) - e_2\right)\right).
\end{aligned}
$$

Since K spans $D \perp$, the above equalities imply that

$$\mathrm{diag}(\nabla^* U(\bar{e}; \lambda))\mathrm{vec}\left((1-t)(\lambda_2^{-1} - 1) + t\left(f_2\left(\frac{u_c(\bar{e}; \lambda)}{\lambda_2}\right) - e_2\right)\right)$$

is in the span of $A^{-1}\mathrm{diag}(\nabla^* U(\bar{e}; \lambda))D$, and hence

$$\begin{aligned}
0 &= \mathrm{vec}\left(\lambda_2^{-1} - (1-t)\right)^\top \mathrm{diag}(\nabla^* U(\bar{e}; \lambda)) \\
&\times \mathrm{vec}\left((1-t)(\lambda_2^{-1} - 1) + t\left(f_2\left(\frac{u_c(\bar{e}; \lambda)}{\lambda_2}\right) - e_2\right)\right).
\end{aligned}$$

On the other hand, we have, for all $\lambda \in \Lambda$

$$0 = \iota^\top \mathrm{diag}(\nabla^* U(\bar{e}; \lambda)) X_{\mathcal{S}(\lambda)} = \iota^\top A^{-1}\mathrm{diag}(\nabla^* U(\bar{e}; \lambda))D.$$

This suggests that

$$0 = \iota^\top \mathrm{diag}(\nabla^* U(\bar{e}; \lambda))\mathrm{vec}\left((1-t)(\lambda_2^{-1} - 1) + t\left(f_2\left(\frac{u_c(\bar{e}; \lambda)}{\lambda_2}\right) - e_2\right)\right),$$

and so we conclude that

$$\begin{aligned}
0 &= \mathrm{vec}\left(\lambda_2^{-1} - 1\right)^\top \mathrm{diag}(\nabla U(\bar{e}; \lambda)) \qquad\qquad\text{(A.1)} \\
&\times \mathrm{vec}\left((1-t)(\lambda_2^{-1} - 1) + t\left(f_2\left(\frac{u_c(\bar{e}; \lambda)}{\lambda_2}\right) - e_2\right)\right)
\end{aligned}$$

holds for all $(\lambda, t) \in H^{-1}(0)$.

Now we claim that (A.1) implies that $H^{-1}(0)$ is bounded. Suppose this is not the case. Then there exists a sequence $\{(\lambda_n, t_n)\} \subset H^{-1}(0)$ such that $|\lambda_n| \to \infty$. However, it is easy to see that the right-hand side of (A.1) diverges as $|\lambda| \to \infty$. This contradicts the equality in (A.1) and establishes the boundedness of $H^{-1}(0)$.

Next, we claim that $H^{-1}(0)$ is closed. In fact, since H is continuous, $H^{-1}(0)$ is closed in the relative topology of $\Lambda \times [0, 1]$, and therefore in order to show that $H^{-1}(0)$ is closed, we only need to prove that the closure of $H^{-1}(0)$ does not meet $\partial\Lambda \times [0, 1]$, where $\partial\Lambda$ denotes the topological boundary of Λ. Suppose that this is not the case. Then there exists a sequence $\{(\lambda_n, t_n)\} \subset H^{-1}(0)$ such that $\lambda_n \to \lambda \in \partial\Lambda$. However, it is again easily verified that the right-hand side of (A.1) diverges as λ approaches $\partial\Lambda$. This violates the equality in (A.1) and proves that $H^{-1}(0)$ is closed.

Since $H^{-1}(0)$ is closed and bounded, it is compact, and hence the system (37) has a solution.

REFERENCES

1 BROWN, D. J., P. M. DEMARZO, AND B. C. EAVES, (1996), Computing equilibria when asset markets are incomplete. *Econometrica* **64**, 1–27.

2 CASS, D., (1984), Competitive equilibria in incomplete financial markets. Working paper, University of Pennsylvania, CARESS.

3 CONSTANTINIDES, G. M., (1982), Intertemporal asset pricing with heterogeneous consumers and without demand aggregation. *Journal of Business* **55**, 253–267.

4 CONZE, A., J.-M. LASRY, AND J. SCHEINKMAN, (1993), Borrowing constraints and international comovements. *Hitotsubashi Journal of Economics* **34**, 23–47.

5 COX, J. C. AND C.-F. HUANG, (1989), Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory* **49**, 33–83.

6 COX, J. C. AND C.-F. HUANG, (1991), A variational problem arising in financial economics. *Journal of Mathematical Economics* **20**, 465–487.

7 DUFFIE, D., (1986), Stochastic equilibria: Existence, spanning number, and the 'No expected financial gain from trade' hypothesis. *Econometrica* **54**, 1161–1183.

Duffie, D., 1987, Stochastic equilibria with incomplete financial markets. *Journal of Economic Theory* **41**, 405–16.

Duffie, D. and C.-f. Huang, 1985, Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities. *Econometrica* **53**, 1337–1356.

Duffie, D. and W. Shafer, 1985, Equilibrium in incomplete markets: I. A basic model of generic existence. *Journal of Mathematical Economics* **14**, 285–300.

Duffie, D. and W. Shafer, 1986, Equilibrium in incomplete markets: II. Generic existence in stochastic economies. *Journal of Mathematical Economics* **15**, 199–216.

Duffie, D. and W. Zame, 1989, The consumption-based capital asset pricing model. *Econometrica* **57**, 1279–1297.

Dumas, B., 1989, Two-person dynamic equilibrium in the capital market. *Review of Financial Studies* **2**, 157-188.

Geanakoplos, J., 1990, An introduction to general equilibrium with incomplete asset markets. *Journal of Mathematical Economics* **19**, 1-38.

Geanakoplos, J. and H. Polemarchakis, 1986, Existence, regularity and constrained suboptimality of competitive allocations when markets are incomplete. In: W. Heller, R. Starr, and D. Starrett (eds.), *Essays in Honor of Kenneth Arrow*, vol. 3, Cambridge University Press, Cambridge.

Geanakoplos, J. and W. Shafer, 1990, Solving systems of simultaneous equations in economics. *Journal of Mathematical Economics* **19**, 69–93.

Girotto, B. and F. Ortu, 1994, Consumption and portfolio policies with incomplete markets and short-sale constraints in the finite-dimensional case: Some remarks. *Mathematical Finance* **4**, 69–73.

Gorman, W., 1953, Community rreference field. *Econometrica* **21**, 63–80.

Hart, O., 1975, On the optimality of equilibrium when the market structure is incomplete. *Journal of Economic Theory* **11**, 418–443.

He, H. and N. D. Pearson, 1991, Consumption and portfolio policies with incomplete markets and short-sale constraints: The finite-dimensional case. *Mathematical Finance* **3**, 1–10.

Huang, C.-f., 1987, An intertemporal general equilibrium asset pricing model: The case of diffusion information. *Econometrica* **55**, 117-142.

Jacka, S. D., 1992, A martingale representation result and an application to incomplete financial markets. *Mathematical Finance* **2**, 239–250.

Judd, K. L., 1998, *Numerical Methods in Economics*. MIT Press, Cambridge.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve, 1987, Optimal portfolio and consumption decisions for a 'small investor' on a finite horizon. *SIAM Journal of Control and Optimization* **25**, 1557–1586.

Karatzas, I., J. P. Lehoczky, S. E. Shreve, and G.-L. Xu, 1991, Martingale and duality methods for utility maximization in an incomplete markets. *SIAM Journal of Control and Optimization* **29**, 702–730.

Karatzas, I., P. Lakner, J. P. Lehoczky, and S. E. Shreve, 1991, Equilibrium in a simplified dynamic, stochastic economy with heterogeneous agents. In: *Stochastic Analysis*, Academic Press, New York.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve, 1990, Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model. *Mathematics of Operations Research* **15**, 80–128.

Kehoe, T. J., 1991, Computation and multiplicity of equilibria. In: Hildebrand, W. and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, vol. IV, Elsevier, New York.

Magill, M. and M. Quinzii, 1996, *Theory of Incomplete Markets, vol. 1*. MIT Press, Cambridge.

Magill, M. and W. Shafer, 1991, Incomplete markets. In: Hildebrand, W.and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, vol. IV, Elsevier, New York.

Negishi, T., 1960, Welfare economics and existence of an equilibrium for a competitive economy. *Metroeconomica* **12**, 92–97.

Ortega, J. M. and W. C. Rheinboldt, 1970, *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York.

Pagès, H., 1987, *Optimal Consumption and Portfolio Policies when Markets are Incomplete*. Mimeo, Massachusetts Institute of Technology.

Pliska, S. R., 1986, A stochastic calculus model of continuous trading: Optimal portfolios. *Mathematics of Operation Research* **11**, 371–382.

Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, 1992, *Numerical Recipes*, 2nd edition. Cambridge University Press, New York.

Radner, R., 1972, Existence of equilibrium of plans, prices and price expectations in a sequence of markets. *Econometrica* **40**, 289–303.

Rubinstein, M., 1974, An aggregation theorem for securities markets. *Journal of Financial Economics* **1**, 225–244.

Werner, J., 1985, Equilibrium in economies with incomplete financial markets. *Journal of Economic Theory* **36**, 110-119.