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Martingale Measure Method for Expected Utility Maximization in Discrete-Time Incomplete Markets

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In this paper we study the expected utility maximization problem for discrete-time incomplete financial markets. As shown by Xia and Yan (2000a, 2000b) in the continuous-time case, this problem can be solved by the martingale measure method. In a special discrete-time model, we explicitly work out the optimal trading strategies and the associated minimum relative entropy martingale measures and minimum Hellinger-Kakutani distance martingale measures.

Key Words: Martingale measure; Incomplete market; Utility maximization; Optimal trading strategy; Relative entropy; Hellinger-Kakutani distance.
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1. INTRODUCTION

If a contingent claim can be replicated by a self-financing trading strategy, then the price of the contingent claim, under the principle of "free of arbitrage opportunities", is the cost of replication. For the problem of maximizing the expected utility of terminal wealth, Karatzas, Lehoczky, and Shreve (1987) and Karatzas, Lehoczky, Shreve and Xu (1991) developed a "replication method". They showed that if some contingent claim that they constructed can be replicated by a self-financing trading strategy whose initial wealth is just the initial capital held by the investor, then the trading strategy is optimal. Thus the notion of replication is important in pricing contingent claims and maximizing the expected utility. In the context of a complete market, any contingent claim can be replicated by a self-financing trading strategy, thus both problems of pricing contingent claims and maximizing the expected utility are well understood. While in an incomplete market, there exist contingent claims that cannot be replicated, thus many troubles appear when dealing with the problems of pricing contingent claims and maximizing the expected utility.

Karatzas, Lehoczky, Shreve and Xu (1991) studied the problem of utility maximization in an incomplete market with a diffusion model. In their model, the market consists of a bond and m stocks, and the price processes of the stocks are driven by a d-dimensional Brownian motion. When $m < d$, that is, the market is incomplete, they augmented the market with certain fictitious stocks so as to create a complete market. Under certain conditions on the model, they showed that one can judiciously choose fictitious stocks such that these fictitious stocks are superfluous in the optimal portfolio of the completed market. In this case, the optimal portfolio of the completed market is also optimal for the original incomplete market.

In general cases, the method of "fictitious completion" is no longer applicable, since the general models are infinite-dimensional ones. From the point of mathematical view, a financial market is arbitrage-free if and only if there exists an equivalent martingale measure for the discounted price processes of the assets and the completeness of the market is equivalent to the uniqueness of the equivalent martingale measure. Thus in incomplete markets there are various equivalent martingale measures. For incomplete markets, Xia and Yan (2000a, 2000b) proposed a martingale measure method to solve the utility maximization problem and unified the so-called "numeraire portfolio approach" and Esscher transform method in the theory of pricing contingent claim. In a geometric Lévy process model, they obtained the associated explicit results. In a discrete-time incomplete market, Schäl (2000a) also studied the connections between martingale measures and portfolio optimization, but the utility functions he considered are HARA utility functions U_γ with $0 \leq \gamma < 1$. In another paper, Schäl

(2000b) studied the relations between arbitrage and utility maximization. The underlying utility function he considered is required to be defined on the positive half-line.

In this paper we consider the same problem as that considered by Xia and Yan (2000a, 2000b), but for the discrete-time case. We first introduce the general results of utility maximization problem for an incomplete market in a general discrete-time setting. For a given utility function, we choose a class of equivalent martingale measures. Corresponding to each of the equivalent martingale measures of this class, we construct a random variable following Karatzas, Lehoczky, Shreve and Xu (1991). If one of these random variables can be replicated by a self-financing trading strategy, then the strategy is optimal and the associated martingale measure is also "optimal" in some sense. For the utility function $\log x$ (resp. $\frac{1}{\gamma}(x^\gamma - 1)(\gamma < 0)$), the associated martingale measure minimizes the relative entropy (resp. the Hellinger-Kakutani distance of order $\frac{\gamma}{\gamma-1}$); for utility function $-e^{-x}$(resp. $-(1-\gamma x)^{\frac{1}{\gamma}}(\gamma < 0)$), the associated martingale measure minimizes the relative entropy (resp. the Hellinger-Kakutani distance of order $\frac{\gamma}{\gamma-1}$) of dual form. For a special discrete-time market model, the optimal trading strategies and the associated "optimal" equivalent martingale measures for the above two classes of utility functions are explicitly worked out.

2. THE GENERAL DISCRETE-TIME MODEL

Let the time index set be $\{0, 1, \cdots, N\}$, and suppose that $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ is a stochastic basis, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $(\mathcal{F}_n)_{0\le n\le N}$ is an increasing complete filtration satisfying $\mathcal{F}_N = \mathcal{F}$. We put $\mathcal{F}_{-1} = \mathcal{F}_0 = \{\emptyset, \Omega\}$.

Assume that in the economy there are one risk-free asset (bond) and d risky assets (stocks) whose price processes are defined as follows:

1) The price of the risk-free asset at time n is $S_n^0, n = 0, 1, \cdots, N$, which is strictly positive and predictable;

2) The price of the i-th risky asset at time n is $S_n^i, i = 1, \cdots, d$. For $i = 1, \cdots, d,\ n = 0, 1, \cdots, N, S_n^i$ is strictly positive and $\mathcal{F}_n$-measurable. We denote $S_n = (S_n^1, \cdots, S_n^d)$ and denote by β_n the discount factor$(S_n^0)^{-1}$ at time n.

A *trading strategy* is a predictable $\mathbb{R}^{d+1}$-valued stochastic sequence $\psi = (\psi_n)_{0\le n\le N},\ \psi_n = (\phi_n^0, \phi_n)$, where $\phi_n = (\phi_n^1, \cdots, \phi_n^d)$, ϕ_n^0 and $\phi_n^i, i = 1, \cdots, d,\ n = 0, 1, \cdots, N$, represent the number of units of the risk-free asset and asset i held at time n respectively. The wealth $V_n(\psi)$ of a trading strategy $\psi = \{\phi^0, \phi\}$ at time n is $V_n(\psi) = \phi_n^0 S_n^0 + \phi_n \cdot S_n$, where $\phi_n \cdot S_n =$

$\sum_{i=1}^{d} \phi_n^i S_n^i$. The discounted wealth is $\widetilde{V}_n(\psi) = \beta_n V_n(\psi) = \phi_n^0 + \phi_n \cdot \widetilde{S}_n$, where $\widetilde{S}_n = (\beta_n S_n^1, \cdots, \beta_n S_n^d)$ is the vector of discounted prices. A trading strategy $\psi = \{\phi^0, \phi\}$ is said to be *self-financing*, if

$$\phi_{n-1}^0 S_{n-1}^0 + \phi_{n-1} \cdot S_{n-1} = \phi_n^0 S_{n-1}^0 + \phi_n \cdot S_{n-1}, \quad \forall 1 \leq n \leq N.$$

It means that at time $n-1$, once the price vector S_{n-1} is quoted, the investor readjusts his/her positions from ψ_{n-1} to ψ_n without bringing or withdrawing any wealth. It is easy to prove that a trading strategy $\psi = \{\phi^0, \phi\}$ is self-financing if and only if

$$\widetilde{V}_n(\psi) = V_0(\psi) + \sum_{k=1}^{n} \phi_k \cdot \Delta \widetilde{S}_k, \quad \forall 1 \leq n \leq N, \tag{1}$$

or equivalently, $\Delta \widetilde{V}_n(\psi) = \phi_n \cdot \Delta \widetilde{S}_n, \quad n \geq 1$, where $\Delta \widetilde{V}_n(\psi) = \widetilde{V}_n(\psi) - \widetilde{V}_{n-1}(\psi)$. For any given $\mathbb{R}^d$-valued predictable process ψ and any constant z, it is easy to construct a unique real-valued predictable process (ϕ_n^0) such that (ϕ_n^0, ϕ_n) is a self-financing strategy with initial wealth z.

A probability measure $\mathbb{Q}$ is called an *equivalent martingale measure*, if it is equivalent to the historical probability measure $\mathbb{P}$ and if the discounted price processes $(\widetilde{S}_n^i)$ of risky assets are $\mathbb{Q}$-martingale. We denote by $\mathcal{P}$ the set of all equivalent martingale measures. For $\mathbb{Q} \in \mathcal{P}$, from (1) we can see that for any self-financing trading strategy ψ, $(\widetilde{V}_n(\psi))$ is a local $\mathbb{Q}$-martingale. It is well-known that there is no arbitrage in the market if and only if $\mathcal{P}$ is not empty. Therefore, we assume that $\mathcal{P}$ is not empty to exclude any arbitrage opportunity. The market is said to be *complete* if $\mathcal{P}$ is a singleton, otherwise we say that the market is *incomplete*.

3. THE UTILITY MAXIMIZATION IN A GENERAL SETTING

3.1. The utility maximization problem for a general utility function

We take the same setting and notations as those in Xia and Yan (2000a, 2000b). Here we only give the main results for easy reference.

We assume that in our model the agent has a utility function $U : (D_U, \infty) \longrightarrow \mathbb{R}$ for wealth, $-\infty \leq D_U < \infty$. Throughout this paper we make the assumption that U is strictly increasing, strictly concave, continuous, and continuously differentiable, and satisfies

$$U'(D_U) \widehat{=} \lim_{x \downarrow D_U} U'(x) = \infty, \quad U'(\infty) \widehat{=} \lim_{x \to \infty} U'(x) = 0.$$

The (continuous, strictly decreasing) inverse of the function U' is denoted by $I:\ (0,\infty)\longrightarrow(D_U,\infty)$.

From the concavity of U we have the following inequality:

$$U(I(y))\geq U(x)+y[I(y)-x],\qquad \forall x>D_U, y>0. \tag{2}$$

We denote by Ψ_s^z the collection of self-financing strategies with initial capital $z>0$. It is easy to show that for any $\psi\in\Psi_s^z,\ \mathbb{Q}\in\mathcal{P},\ (\widetilde{V}_n(\psi))$ is a $\mathbb{Q}$-local martingale.

We will discuss the cases of $D_U>-\infty$ and $D_U=-\infty$, respectively.

In the case of $D_U>-\infty$, for a given initial capital $z>0$, we denote

$$\Psi_s^z(D_U)\widehat{=}\{\psi:\psi\in\Psi_s^z, V_N(\psi)>D_U\}.$$

We consider the problem of maximizing the expected utility of terminal wealth $\mathbb{E}\left[U(V_T(\psi))\right]$, over the class $\Psi_s^z(D_U)$. A strategy $\psi\in\Psi_s^z(D_U)$ which maximizes the expected utility is called optimal. It is easy to show that the wealth process of optimal trading strategies in $\Psi_s^z(D_U)$ is unique.

For $\mathbb{Q}\in\mathcal{P}$, we denote $Z_n^{\mathbb{Q}}\widehat{=}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_n\right]$, then $(Z_n^{\mathbb{Q}})$ is a strict positive martingale. Put

$$\mathcal{P}_n\widehat{=}\{\ \mathbb{Q}\in\mathcal{P}:\ |\mathbb{E}[\beta_n Z_n^{\mathbb{Q}} I(y\beta_n Z_n^{Q})]|<\infty,\ \forall y\in(0,\infty)\},\quad n=0,1,\cdots,N,$$

and we make the standing assumption that $\mathcal{P}_n$ is not empty. The following notion was initiated by Karatzas, Lehoczky, Shreve and Xu (1991). For every $n=0,1,\cdots,N$ and $\mathbb{Q}\in\mathcal{P}_n$, the function $\mathcal{X}_n^{\mathbb{Q}}:(0,\infty)\longrightarrow(D_U\mathbb{E}_{\mathbb{Q}}[\beta_n],\infty)$, defined by

$$\mathcal{X}_n^{\mathbb{Q}}(y)\widehat{=}\mathbb{E}\left[\beta_n Z_n^{\mathbb{Q}} I(y\beta_n Z_n^{\mathbb{Q}})\right],\quad 0<y<\infty$$

inherits from I the property of being a continuous, strictly decreasing mapping from $(0,\infty)$ onto $(D_U\mathbb{E}_{\mathbb{Q}}[\beta_n],\infty)$, and hence $\mathcal{X}_n^{\mathbb{Q}}$ has a (continuous, strictly decreasing) inverse $\mathcal{Y}_n^{\mathbb{Q}}$ from $(D_U\mathbb{E}_{\mathbb{Q}}[\beta_n],\infty)$ onto $(0,\infty)$. We define

$$\xi_n^{\mathbb{Q}}(x)\widehat{=}I(\mathcal{Y}_n^{\mathbb{Q}}(x)\beta_n Z_n^{\mathbb{Q}}),\quad 0\leq n\leq N,\ \mathbb{Q}\in\mathcal{P}_n,\ x\in(D_U\mathbb{E}_{\mathbb{Q}}[\beta_n],\infty). \tag{3}$$

We can see that for $\psi\in\Psi_s^z(D_U),\mathbb{Q}\in\mathcal{P},\ (\beta_n V_n(\psi))$ is a $\mathbb{Q}$-martingale. Associated with (2) and (3) we have

$$\mathbb{E}\left[U(\xi_N^{\mathbb{Q}}(z))\right]\geq\mathbb{E}\left[U(V_N(\psi))\right],\mathbb{Q}\in\mathcal{P}_N,\ \psi\in\Psi_s^z(D_U). \tag{4}$$

From the discussion above we know that if there exists a probability measure $\mathbb{Q}^* \in \mathcal{P}_N$ and a trading strategy $\widehat{\psi} \in \Psi_s^z(D_U)$ such that $\xi_N^{\mathbb{Q}^*}(x) = V_N(\widehat{\psi})$, then $\widehat{\psi}$ is optimal. Since $Z_N^{\mathbb{Q}}$ is uniquely determined by $\xi_N^{\mathbb{Q}}(x)$, such a $\mathbb{Q}^*$ is unique. From (4) we see that $\mathbb{Q}^*$ is also "optimal" in $\mathcal{P}_N$ in the sense that

(**A**). $\mathbb{E}\left[U(\xi_N^{\mathbb{Q}^*}(x))\right] \le \mathbb{E}\left[U(\xi_N^{\mathbb{Q}}(x))\right], \forall \mathbb{Q} \in \mathcal{P}_N$.

In the case of $D_U = -\infty$, put

$$\mathcal{P}' \widehat{=} \{\mathbb{Q} \in \mathcal{P} : \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\Omega, \mathcal{F}_N, \mathbb{P})\},$$

$$\widehat{\Psi}_s^z \widehat{=} \{\psi : \psi \in \Psi_s^z,\ \beta_n V_n(\psi) \in L^2(\Omega, \mathcal{F}_n, \mathbb{P}),\quad 0 \le n \le N\},$$

and assume that $\mathcal{P}'$ is not empty. For $\psi \in \widehat{\Psi}_s^z, \mathbb{Q} \in \mathcal{P}'$, it can be shown that $(\beta_n V_n(\psi))$ is a $\mathbb{Q}$-martingale. For the case of $D_U = -\infty$, we consider the martingale measure in $\mathcal{P}'$ instead of $\mathcal{P}$ and replace the notion $\mathcal{P}_n$ corresponding to $\mathcal{P}$ by $\mathcal{P}'_n$ and replace $\Psi_s^z(D_U)$ with $\widehat{\Psi}_s^z$. By the same way as in the case of $D_U > -\infty$, we can see that if there exists a probability measure $\mathbb{Q}^* \in \mathcal{P}'_N$ and a trading strategy $\widehat{\psi} \in \widehat{\Psi}_s^z$ such that $\xi_N^{\mathbb{Q}^*}(z) = V_N(\widehat{\psi})$, then $\widehat{\psi}$ is optimal over $\widehat{\Psi}_s^z$ and the measure $\mathbb{Q}$ is "optimal" over $\mathcal{P}'_N$.

3.2. The HARA utility maximization, the minimum relative entropy and the minimum Hellinger-Kakutani distance

In the sequel, we assume that (S_n^0) is deterministic, then so is (β_n). Here we consider the HARA (hyperbolic absolute risk aversion) utility functions given by

$$U_\gamma(x) = \begin{cases} \frac{1}{\gamma}(x^\gamma - 1), & \gamma < 0, \\ \log x, & \gamma = 0. \end{cases}$$

For the HARA utility functions $U = U_\gamma(x)(\gamma \le 0)$, we have $D_U = 0, I(x) = x^{\frac{1}{\gamma-1}}$ and $\mathcal{P}_n = \mathcal{P}$ for $n = 0, 1, \cdots, N$. Put $\delta \widehat{=} \frac{\gamma}{\gamma-1} \in [0, 1)$, then $\frac{1}{\delta} + \frac{1}{\gamma} = 1$ for $\gamma < 0$. From (3) we have

$$\xi_n^{\mathbb{Q}}(x) = \frac{x(Z_n^{\mathbb{Q}})^{\frac{1}{\gamma-1}}}{\beta_n \mathbb{E}[(Z_n^{\mathbb{Q}})^\delta]}, \quad n = 0, 1, \cdots, N. \tag{5}$$

Then for the HARA utility functions $U = U_\gamma(\gamma \le 0)$, the statement (A) in Section 3.1 is equivalent to the following statement (B)(resp. (C)) when $\gamma < 0$ (resp. $\gamma = 0$):

(**B**). $\mathbb{E}\left[(Z_N^{\mathbb{Q}^*})^\delta\right] \ge \mathbb{E}\left[(Z_N^{\mathbb{Q}})^\delta\right], \forall \mathbb{Q} \in \mathcal{P}$.

(**C**). $\mathbb{E}[\log Z_N^{\mathbb{Q}^*}] \ge \mathbb{E}[\log Z_N^{\mathbb{Q}}], \forall \mathbb{Q} \in \mathcal{P}$.

DEFINITION 3.1. 1). Assume that probability measure $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ and $\delta \in (0,1)$. Put

$$H_\delta(\mathbb{Q},\mathbb{P}) \widehat{=} \mathbb{E}_\mathbb{P}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^\delta\right] = \mathbb{E}_\mathbb{P}\left[(Z_N^\mathbb{Q})^\delta\right], \quad d_\delta(\mathbb{Q},\mathbb{P}) \widehat{=} 2(1 - H_\delta(\mathbb{Q},\mathbb{P})).$$

$H_\delta(\mathbb{Q},\mathbb{P})$ and $d_\delta(\mathbb{Q},\mathbb{P})$ are called the *Hellinger integral* and *Hellinger-Kakutani distance* of order δ of $\mathbb{P}$ with respect to $\mathbb{Q}$, respectively;

2). The *relative entropy* of a probability measure $\mathbb{P}$ with respect to $\mathbb{Q}$ is defined by

$$I_\mathbb{Q}(\mathbb{P}) = \mathbb{E}_\mathbb{Q}\left[\frac{d\mathbb{P}}{d\mathbb{Q}} \log \frac{d\mathbb{P}}{d\mathbb{Q}}\right] = -\mathbb{E}_\mathbb{P}\left[\log \frac{d\mathbb{Q}}{d\mathbb{P}}\right].$$

Both the Hellinger-Kakutani distance and the relative entropy are quantitative measures of the difference between $\mathbb{Q}$ and $\mathbb{P}$. One should be aware that in general $d_\delta(\mathbb{Q},\mathbb{P})$(resp. $I_\mathbb{Q}(\mathbb{P})$) is not equal to $d_\delta(\mathbb{P},\mathbb{Q})$(resp. $I_\mathbb{P}(\mathbb{Q})$) and it is easy to show that for $\delta \in (0,1)$, $d_\delta(\mathbb{Q},\mathbb{P}) = d_{1-\delta}(\mathbb{P},\mathbb{Q})$.

For the HARA utility functions $U = U_\gamma(\gamma \le 0)$, by the results in Subsection 3.1, we know that if there exist a probability measure $\mathbb{Q}^* \in \mathcal{P}$ and a strategy $\widehat{\psi} \in \Psi_s^z(D_U)$ such that $\xi_N^{\mathbb{Q}^*}(x) = V_N(\widehat{\psi})$, then $\widehat{\psi}$ is optimal and

i) for $U(x) = U_\gamma(\gamma < 0)$, the Hellinger-Kakutani distance of order δ of the historical measure $\mathbb{P}$ with respect to $\mathbb{Q}^*$ is the minimum over $\mathcal{P}$, that is, $d_\delta(\mathbb{Q}^*,\mathbb{P}) = \min\limits_{\mathbb{Q}\in\mathcal{P}} d_\delta(\mathbb{Q},\mathbb{P})$, where δ satisfies $\frac{1}{\delta} + \frac{1}{\gamma} = 1$;

ii) for $U(x) = \log x$, the relative entropy of $\mathbb{P}$ with respect to $\mathbb{Q}^*$ is the minimum over $\mathcal{P}$, that is $I_{\mathbb{Q}^*}(\mathbb{P}) = \min\limits_{\mathbb{Q}\in\mathcal{P}} I_\mathbb{Q}(\mathbb{P})$.

3.3. Results of dual form

In the following we consider the class of utility functions given by

$$W_\gamma(x) = \begin{cases} -(1-\gamma x)^{\frac{1}{\gamma}}, & \gamma < 0, \\ -e^{-x}, & \gamma = 0. \end{cases}$$

Since $U_\gamma(-W_\gamma(x)) = -x(\gamma \le 0)$, we say that $W_\gamma(x)$ is the dual utility function of $U_\gamma(x)$. For $W_\gamma(x)(\gamma \le 0)$ we have

$$D_U = \begin{cases} \frac{1}{\gamma}, & \gamma < 0 \\ -\infty, & \gamma = 0 \end{cases}, \quad I(x) = \begin{cases} \frac{1 - x^{\frac{\gamma}{1-\gamma}}}{\gamma}, & \gamma < 0 \\ -\log x, & \gamma = 0. \end{cases}$$

When $\gamma < 0$, we have $\mathcal{P}_n = \mathcal{P}$ for $n = 0, 1, \cdots, N$ and when $\gamma = 0$, $\mathcal{P}_N = \{\mathbb{Q} \in \mathcal{P} : |I_\mathbb{P}(\mathbb{Q})| < \infty\}$. Put $\delta \widehat{=} \frac{\gamma}{\gamma-1}$, then $\frac{1}{\delta} + \frac{1}{\gamma} = 1$ and $\delta \in [0,1)$. In this subsection we replace the notation $\xi_n^\mathbb{Q}(x)$ with $\zeta_n^\mathbb{Q}(x)$ (resp. $\eta_n^\mathbb{Q}(x)$) when

$\gamma < 0$ (resp. $\gamma = 0$). Thus from (3) we have

$$\zeta_n^{\mathbb{Q}}(x) = \frac{1}{\gamma}\left\{1 - \frac{\beta_n - \gamma x}{\beta_n}\frac{(Z_n^{\mathbb{Q}})^{\frac{\gamma}{1-\gamma}}}{\mathbb{E}\left[(Z_n^{\mathbb{Q}})^{\frac{1}{1-\gamma}}\right]}\right\}, \tag{6}$$

$$\eta_n^{\mathbb{Q}}(x) = \frac{x}{\beta_n} + \mathbb{E}[Z_n^{\mathbb{Q}} \log Z_n^{\mathbb{Q}}] - \log Z_n^{\mathbb{Q}}. \tag{7}$$

Then for utility functions $W_\gamma(x)$, the statement (A) above is equivalent to the following statement (D)(resp. (E)) when $\gamma < 0$ (resp. $\gamma = 0$):

(**D**). $\mathbb{E}\left[(Z_N^{\mathbb{Q}^*})^{\frac{1}{1-\gamma}}\right] \geq \mathbb{E}\left[(Z_N^{\mathbb{Q}})^{\frac{1}{1-\gamma}}\right]$, $\forall\, \mathbb{Q} \in \mathcal{P}$,

(**E**). $\mathbb{E}[Z_N^{\mathbb{Q}^*} \log Z_N^{\mathbb{Q}^*}] \leq \mathbb{E}[Z_N^{\mathbb{Q}} \log Z_N^{\mathbb{Q}}], \forall\, \mathbb{Q} \in \mathcal{P}_N$.

By the results in Subsection 3.1 we know that if there exist a probability measure $\mathbb{Q}_1^* \in \mathcal{P}$ and a strategy $\widehat{\psi}_1 \in \Psi_s^z(D_U)$ such that $\zeta_N^{\mathbb{Q}_1^*}(z) = V_N(\widehat{\psi}_1)$, then $\widehat{\psi}_1$ is optimal for $W_\gamma(\gamma < 0)$ and the Hellinger-Kakutani distance of order δ of $\mathbb{Q}_1^*$ with respect to $\mathbb{P}$ is the minimum over $\mathcal{P}$, that is, $d_\delta(\mathbb{P}, \mathbb{Q}_1^*) = \min_{\mathbb{Q}\in\mathcal{P}} d_\delta(\mathbb{P}, \mathbb{Q})$, where δ satisfies $\frac{1}{\delta} + \frac{1}{\gamma} = 1$; if there exist a probability measure $\mathbb{Q}_2^* \in \mathcal{P}'$ with finite relative entropy $I_{\mathbb{P}}(\mathbb{Q}_2^*)$ and a strategy $\widehat{\psi}_2 \in \widehat{\Psi}_s^z$ such that $\xi_N^{\mathbb{Q}_2^*}(z) = V_N(\widehat{\psi}_2)$, then $\widehat{\psi}_2$ is optimal for $W_0 = -e^{-x}$ and the relative entropy of $\mathbb{Q}_2^*$ with respect to $\mathbb{P}$ is the minimum over $\mathcal{P}'_N$, that is $I_{\mathbb{P}}(\mathbb{Q}_2^*) = \min_{\mathbb{Q}\in\mathcal{P}} I_{\mathbb{P}}(\mathbb{Q})$.

4. THE UTILITY MAXIMIZATION PROBLEM FOR A SPECIAL MARKET MODEL

4.1. The market model and the characterization of equivalent martingale measures

In this section we consider a discrete-time incomplete financial market in which there are only two assets: one risk-free asset (bond) and one risky asset (stock) whose price processes S_n^0 and S_n satisfy:

1) $S_0^0 = 1, S_0$ is a positive constant;

2) $S_n^0 = S_{n-1}^0(1 + r_n)$ and r_n is a positive constant, for $n = 1, \cdots, N$;

3) $\forall n = 1, \cdots, N, S_n = S_{n-1}(1 + R_n)$, where R_n is an $\mathcal{F}_n$-measurable random variable independent of $\mathcal{F}_{n-1}$, and $R_n > -1$, a.s..

Assume that for each $n = 1, \cdots, N, \mathcal{F}_n = \sigma(R_1, \cdots, R_n), \mathcal{F}_0 = \{\emptyset, \Omega\}$. For notational convenience, we put $\mathcal{F}_{-1} = \mathcal{F}_0$.

We assume that for each n, there exist two constants d_n and u_n such that $d_n < r_n < u_n, \mathbb{P}(d_n \leq R_n \leq u_n) = 1$, and for any $\varepsilon > 0, \mathbb{P}(R_n \geq d_n - \varepsilon) > 0$ and $\mathbb{P}(R_n \leq u_n - \varepsilon) > 0$. Consequently, the support of the distribution of

R_n is $[d_n, u_n]$. Readers can refer to Li and Yan (1999) for the economic meaning of the above conditions.

From the above 2) and 3) we have

$$S_n^0 = \prod_{k=1}^n (1 + r_k), \qquad S_n = S_0 \prod_{k=1}^n (1 + R_k), \quad \forall n = 1, \cdots, N. \tag{8}$$

Recalling that $\beta_n = (S_n^0)^{-1}$, we have

$$\widetilde{S}_n = \beta_n S_n,\ \Delta \widetilde{S}_n = \widetilde{S}_n - \widetilde{S}_{n-1} = \widetilde{S}_{n-1}\left(\frac{1+R_n}{1+r_n} - 1\right) = \widetilde{S}_{n-1}\frac{R_n - r_n}{1+r_n}.$$

For a self-financing trading strategy $\psi = \{\phi^0, \phi\}$, we put

$$\pi_n^0 = \frac{\phi_n^0 S_{n-1}^0}{V_{n-1}}, \quad \pi_n = \frac{\phi_n S_{n-1}}{V_{n-1}}, \quad n = 1, \cdots, N.$$

Then π_n^0 and π_n represent the proportion of the wealth V_{n-1} invested in the risk-free asset and the risky asset at time n, respectively. Since

$$V_{n-1} = \phi_{n-1}^0 S_{n-1}^0 + \phi_{n-1} S_{n-1} = \phi_n^0 S_{n-1}^0 + \phi_n S_{n-1},$$

we have $\pi_n^0 = 1 - \pi_n$. We also call (π_n^0, π_n) a *portfolio* at time n. Thus

$$\widetilde{V}_n - \widetilde{V}_{n-1} = \Delta \widetilde{V}_n = \phi_n \Delta \widetilde{S}_n = \phi_n \widetilde{S}_{n-1}\frac{R_n - r_n}{1+r_n} = \pi_n \widetilde{V}_{n-1}\frac{R_n - r_n}{1+r_n}.$$

Consequently,

$$V_n = V_{n-1}[1 + r_n + \pi_n(R_n - r_n)].$$

When π_n takes value in $\left(-\frac{1+r_n}{u_n - r_n}, \frac{1+r_n}{r_n - d_n}\right)$, $V_n(\psi)$ is strictly positive.

We denote by $(V_n^{z,\pi})$ the solution of the equation

$$\Delta \widetilde{V}_n^{z,\pi} = \pi_n \widetilde{V}_{n-1}^{z,\pi}\frac{R_n - r_n}{1+r_n}, \quad V_0^{z,\pi} = z.$$

Then $V_n^{z,\pi} = V_n(\psi) = z \prod_{k=1}^n [1 + r_k + \pi_k(R_k - r_k)]$.

From (8) we have

$$\widetilde{S}_n = \beta_n S_n = S_0 \prod_{k=1}^n \frac{1+R_k}{1+r_k} = S_0 \prod_{k=1}^n \left(1 + \frac{R_k - r_k}{1+r_k}\right). \tag{9}$$

Let (M_n) be a local martingale with $M_0 = 0$. Since $M_n \in \mathcal{F}_n$, from the Doob Representation Theorem we know that there exists an n-variate Borel-measurable function $g_n(x_1, \cdots, x_n)$ such that

$$M_n = g_n\left(\frac{R_1 - r_1}{1 + r_1}, \cdots, \frac{R_n - r_n}{1 + r_n}\right), n = 1, \cdots, N. \tag{10}$$

From $\mathbb{E}(\Delta M_n | \mathcal{F}_{n-1}) = 0$ we know that

$$M_{n-1} = \mathbb{E}\left[g_n\left(\frac{R_1 - r_1}{1 + r_1}, \cdots, \frac{R_n - r_n}{1 + r_n}\right)\middle| \mathcal{F}_{n-1}\right], \qquad n = 1, \cdots, N.$$

Therefore,

$$\begin{aligned} M_n \; = \; & \sum_{k=1}^{n}\left[g_k\left(\frac{R_1 - r_1}{1 + r_1}, \cdots, \frac{R_k - r_k}{1 + r_k}\right)\right. \\ - \; & \left.\mathbb{E}\left(g_k\left(\frac{R_1 - r_1}{1 + r_1}, \cdots, \frac{R_k - r_k}{1 + r_k}\right)\middle| \mathcal{F}_{k-1}\right)\right]. \end{aligned} \tag{11}$$

If $\mathbb{Q}$ is a probability measure and is equivalent to $\mathbb{P}$, we put $Z_n \widehat{=} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle| \mathcal{F}_n\right]$, then (Z_n) is a strictly positive martingale. Put $M_n = \sum_{k=1}^{n} \frac{\Delta Z_k}{Z_{k-1}}$, then (M_n) is a local martingale and $Z_n = \prod_{k=1}^{n}(1 + \Delta M_k)$. From the above analysis we know that there exists an n-variate Borel-measurable function $g_n(x_1, \cdots, x_n)$ such that (10) holds. Hence the density process $(Z_n)_{1\le n\le N}$ associated with $g = (g_n)_{1\le n\le N}$ is

$$\begin{aligned} Z_g(n) \; \widehat{=} \; & \prod_{k=1}^{n}\left[1 + g_k\left(\tfrac{R_1 - r_1}{1 + r_1}, \cdots, \tfrac{R_k - r_k}{1 + r_k}\right)\right. \\ & \left. - \, \mathbb{E}\left(g_k\left(\tfrac{R_1 - r_1}{1 + r_1}, \cdots, \tfrac{R_k - r_k}{1 + r_k}\right)\middle| \mathcal{F}_{k-1}\right)\right], \quad 1 \le n \le N. \end{aligned} \tag{12}$$

Specifically, if $\mathbb{Q}$ is an equivalent martingale measure, that is, $(\widetilde{S}_n)$ is a $\mathbb{Q}$-martingale, then $\mathbb{E}_{\mathbb{Q}}(R_n | \mathcal{F}_{n-1}) = r_n$. Thus $\mathbb{E}[Z_g(n)(R_n - r_n)|\mathcal{F}_{n-1}] = 0$. From (12) we have

$$\begin{aligned} & Z_g(n-1)\mathbb{E}\left[(R_n - r_n)\left(1 + g_n\left(\frac{R_1 - r_1}{1 + r_1}, \cdots, \frac{R_n - r_n}{1 + r_n}\right)\right.\right. \\ & \left.\left. - \, \mathbb{E}\left(g_n\left(\frac{R_1 - r_1}{1 + r_1}, \cdots, \frac{R_n - r_n}{1 + r_n}\right)\middle| \mathcal{F}_{n-1}\right)\right)\middle| \mathcal{F}_{n-1}\right] = 0. \end{aligned}$$

That is,

$$\begin{aligned}&\mathbb{E}\Big[(R_n - r_n)\Big(1 + g_n\left(\tfrac{R_1-r_1}{1+r_1}, \cdots, \tfrac{R_n-r_n}{1+r_n}\right)\\ &- \mathbb{E}\left[g_n\left(\tfrac{R_1-r_1}{1+r_1}, \cdots, \tfrac{R_n-r_n}{1+r_n}\right)\Big| \mathcal{F}_{n-1}\right]\Big)\Big| \mathcal{F}_{n-1}\Big] = 0.\end{aligned} \tag{13}$$

In the following we denote by $\mathbb{G}$ the collection of all $g \widehat{=} (g_n)_{1\le n\le N}$, where g_n is an n-variate Borel-measurable function and $(Z_g(n))_{1\le n\le N}$ is a strictly positive martingale and denote the probability measure associated with $g \in \mathbb{G}$ by $\mathbb{Q}_g$, that is, $Z_g(n) = \mathbb{E}\left(\frac{d\mathbb{Q}_g}{d\mathbb{P}}\Big| \mathcal{F}_n\right)$. The random variable $\xi_n^{\mathbb{Q}_g}$ is denoted for short by ξ_n^g. Then from the above analysis we obtain the following theorem which gives a characterization for equivalent martingale measures.

THEOREM 4.1. *A probability measure $\mathbb{Q}$ is an equivalent martingale measure if and only if $\mathbb{Q} = \mathbb{Q}_g$ for some $g \in \mathbb{G}$ such that (13) holds.*

4.2. The HARA Utility Maximization for the Special Model

In this subsection, we will explicitly work out the optimal trading strategy, the minimum relative entropy martingale measure and the minimum Hellinger-Kakutani distance martingale measure for the HARA utility functions $U = U_\gamma(x)(\gamma \le 0)$.

LEMMA 4.1. *For $n = 1, \cdots, N$, let F_n be the distribution function of R_n and assume that $\int_{\mathbb{R}} |x|F_n(dx) < \infty$. For given $\gamma \le 0$ we put*

$$f_n(a) = \int_{\mathbb{R}} \frac{x - r_n}{(1 + r_n + a(x - r_n))^{1-\gamma}} F_n(dx), \quad a \in \left(-\frac{1+r_n}{u_n - r_n}, \frac{1+r_n}{r_n - d_n}\right). \tag{14}$$

If $\mathbb{E}[R_n] = r_n$, then $f_n(a) = 0$ has a unique solution $\pi_n^* = 0$. In other case, $f_n(a) = 0$ has a unique solution π_n^* in $(-\frac{1+r_n}{u_n-r_n}, \frac{1+r_n}{r_n-d_n})$ if and only if

$$\lim_{a\downarrow -\frac{1+r_n}{u_n-r_n}} f_n(a) > 0, \qquad \lim_{a\uparrow \frac{1+r_n}{r_n-d_n}} f_n(a) < 0. \tag{15}$$

Proof. See the appendix. ∎

In the following we assume that (15) holds and $\pi_n^* \in \left(-\frac{1+r_n}{u_n-r_n}, \frac{1+r_n}{r_n-d_n}\right)$ is the unique solution of the equation $f_n(a) = 0$, (V_n^{z,π^*}) is the wealth process corresponding to π^*. Then $V_n^{z,\pi^*} = z \prod_{k=1}^{n} [1 + r_k + \pi_k^*(R_k - r_k)]$.

THEOREM 4.2. *Let*

$$g_n^*(x_1, \cdots, x_n) = \frac{(1+\pi_n^* x_n)^{\gamma-1}}{\mathbb{E}[(1+\pi_n^* x_n)^{\gamma-1}]}, \quad 1 \le n \le N.$$

Then $g^* \in \mathbb{G}, \mathbb{Q}_{g^*} \in \mathcal{P}$ *and* $\xi_N^{g^*}(z) = V_N^{z,\pi^*}$ *for the utility function* $U_\gamma(x)(\gamma \le 0)$.

Proof. See the appendix. ∎

By Theorem 4.2 and the results in Subsection 3.2 we know that the strategy corresponding to π^* is optimal for the HARA utility functions $U = U_\gamma(\gamma \le 0)$ and $\mathbb{Q}_{g^*}$ is just the associated optimal martingale measure. That is, when $\gamma < 0, \mathbb{Q}_{g^*}$ minimizes the Hellinger-Kakutani distance (of order δ) $d_\delta(\mathbb{Q}, \mathbb{P})$ over $\mathcal{P}$, where δ satisfies $\frac{1}{\delta} + \frac{1}{\gamma} = 1$; when $\gamma = 0$, $\mathbb{Q}_{g^*}$ minimizes the relative entropy $I_{\mathbb{Q}}(\mathbb{P})$ over $\mathcal{P}$.

4.3. Results of dual form

In this subsection we will work out the optimal strategies, the minimum relative entropy martingale measure and the minimum Hellinger-Kakutani distance martingale measure of dual form for the utility function $W_{\gamma'}(\gamma' \le 0)$ given by

$$W'_\gamma(x) = \begin{cases} -(1-\gamma' x)^{\frac{1}{\gamma'}}, & \gamma' < 0, \\ -e^{-x}, & \gamma' = 0. \end{cases}$$

We will first consider the case of $\gamma' < 0$. In this case, $D_U = \frac{1}{\gamma'}$. Put $\delta' = \frac{\gamma'}{\gamma'-1}$, then $\frac{1}{\delta'} + \frac{1}{\gamma'} = 0$. Let $\gamma = \frac{1}{\gamma'}, \frac{1}{\delta} + \frac{1}{\gamma} = 1$, then $\delta = \frac{\gamma}{\gamma-1} = 1 - \delta'$. Under the notation of Section 3.3 and Section 4.1 and the condition of Theorem 4.2 we put

$$\zeta_n^{g^*}(x) = \frac{1}{\gamma'} \left\{ 1 - \frac{\beta_n - \gamma' x}{\beta_n} \cdot \frac{(Z_{g^*}(n))^{\frac{\gamma'}{1-\gamma'}}}{\mathbb{E}[(Z_{\widehat{g}(n)})^{\frac{1}{1-\gamma'}}]} \right\}.$$

Then from Theorem 4.2 we have

$$\frac{(Z_{g^*}(n))^{\frac{\gamma'}{1-\gamma'}}}{\mathbb{E}\left[(Z_{g^*(n)})^{\frac{1}{1-\gamma'}} \right]} = \frac{(Z_{g^*}(n))^{\frac{1}{\gamma-1}}}{\mathbb{E}\left[(Z_{g^*}(n))^{\delta} \right]} = \frac{\beta_n \xi_n^{g^*}(z)}{z} = \frac{\beta_n V_n^{z,\pi^*}}{z}.$$

Thus

$$\zeta_n^{g^*}(z) = \frac{1}{\gamma'} + \left(1 - \frac{\beta_n}{\gamma' z} \right) V_n^{z,\pi^*}.$$

Now we construct self-financing strategy $\widehat{\psi} = \{\widehat{\phi}^0, \widehat{\phi}\}$ defined as follows: let $\psi = (\phi^0, \phi)$ be the strategy corresponding to the portfolio π^*, we put $\widehat{\phi}_n^0 = \left(1 - \frac{\beta_N}{\gamma' z}\right)\phi_n^0 + \frac{\beta_N}{\gamma'}$, $\widehat{\phi}_n = (1 - \frac{\beta_N}{\gamma' z})\phi_n$. Then

$$\widetilde{V}_n(\widehat{\psi}) \;=\; \frac{\beta_N}{\gamma'} + \beta_n \left(1 - \frac{\beta_N}{\gamma' z}\right) V_n^{z,\pi^*}, \quad n = 1, \cdots, N.$$

Specifically

$$V_N(\widehat{\psi}) = \frac{1}{\gamma'} + \left(1 - \frac{\beta_N}{\gamma' z}\right) V_N^{z,\pi^*}.$$

Recalling that $V_n^{z,\pi^*} > 0$, thus $\widetilde{V}_n(\widehat{\psi}) > \frac{1}{\gamma'}$. Consequently, $\widehat{\psi} \in \Psi_s^z(D_U)$ and $V_N(\widehat{\psi}) = \zeta_N^{g^*}(z)$. Then from the results in Section 3.3 we know that $\widehat{\psi}$ is optimal for $W'_\gamma(x) = -(1-\gamma' x)^{\frac{1}{\gamma'}} (\gamma' < 0)$ and the martingale measure $\mathbb{Q}_{g^*}$ minimizes the Hellinger-Kakutani distance (of order δ') $d_{\delta'}(\mathbb{P}, \mathbb{Q})$ over $\mathcal{P}$.

For utility function $W_0(x) = -e^{-x}$, we have $D_U = -\infty$. And for $g = (g_n) \in \mathbb{G}$,

$$\eta_n^g(x) = \frac{x}{\beta_n} + \mathbb{E}[Z_g(n) \log Z_g(n)] - \log Z_g(n).$$

Lemma 4.2. *For $n = 1, \cdots, N$, let F_n be the distribution function of R_n and assume that*
$\int_{\mathbb{R}} |x| F_n(dx) < \infty$. Put

$$h_n(a) = \int_{\mathbb{R}} (x - r_n) e^{-a\frac{x-r_n}{1+r_n}} F_n(dx), \quad 1 \le n \le N \tag{16}$$

Then there exists some $b_n(-\infty \le b_n \le 0)$ such that $h_n(a)$ is $+\infty$ for $a \le b_n$ and finite for $a > b_n$. Furthermore, $h_n(a) = 0$ has a unique solution $\widehat{\pi}_n$ in (b_n, ∞) if and only if $\lim_{a \downarrow b_n} h_n(a) > 0$.

Proof. See the appendix. ▮

In the following we assume that $\lim_{a \downarrow b_n} h_n(a) > 0$, for $n = 1, \cdots, N$. Then $h_n(a) = 0$ has a unique solution $\widehat{\pi}_n \in (b_n, \infty)$. For $n = 1, \cdots, N$, define $\phi_k^n = \frac{\beta_n \widehat{\pi}_k}{\beta_{k-1} S_{k-1}}$, $k = 1, \cdots, n$, and denote by ψ^n the corresponding self-financing strategy with the initial wealth z. Then the discounted wealth of

ψ^n is

$$\begin{aligned}\widetilde{V}_n(\psi^n) &= z + \sum_{k=1}^{n} \phi_k^n \Delta \widetilde{S}_k = z + \sum_{k=1}^{n} \phi_k^n \beta_{k-1} S_{k-1} \frac{R_k - r_k}{1 + r_k} \\ &= z + \sum_{k=1}^{n} \beta_n \widehat{\pi}_k \frac{R_k - r_k}{1 + r_k} = z + \beta_n \sum_{k=1}^{n} \widehat{\pi}_k \frac{R_k - r_k}{1 + r_k}.\end{aligned}$$

Thus

$$V_n(\psi^n) = \frac{z}{\beta_n} + \sum_{k=1}^{n} \widehat{\pi}_k \frac{R_k - r_k}{1 + r_k}.$$

THEOREM 4.3. *Let*

$$\widehat{g}_n(x_1, \cdots, x_n) = \frac{e^{-\widehat{\pi}_n x_n}}{\mathbb{E}[e^{-\widehat{\pi}_n x_n}]}, \quad 1 \le n \le N.$$

Then $\widehat{g} \in \mathbb{G}, \mathbb{Q}_{\widehat{g}} \in \mathcal{P}'$ *and* $\psi^N \in \widehat{\Psi}_s^z$, $\eta_N^{\widehat{g}}(z) = V_N(\psi^N)$ *for the utility function* $W_0(x) = -e^{-x}$.

Proof. See the appendix. ▌

By Theorem 4.3 and the results in Section 3.3 we know that the strategy ψ^N is optimal for the utility function $-e^{-x}$ and $\mathbb{Q}_{\widehat{g}}$ is just the equivalent martingale measure minimizing the relative entropy $I_{\mathbb{P}}(\mathbb{Q})$ over $\mathcal{P}'_N$.

5. CONCLUSION

For the problem of maximizing the expected utility of terminal wealth, Karatzas, Lehoczky, and Shreve (1987) and Karatzas, Lehoczky, Shreve and Xu (1991) developed a "replication method" and gave satisfactory results for complete markets. For an incomplete market, Karatzas, Lehoczky, Shreve and Xu (1991) studied this problem in a market with a diffusion model using the method of "fictitious completion". In their model, the market consists of a bond and m stocks, and the price processes of the stocks are driven by a d-dimensional Brownian motion. When $m < d$, that is, the market is incomplete, they augmented the market with certain fictitious stocks so as to create a complete market and showed that the optimal portfolio of the completed market is also optimal for the original incomplete market. But for general infinite-dimensional market models, the "fictitious completion" method is no longer applicable. For continuous-time incomplete markets, Xia and Yan (2000a, 2000b) proposed a martingale measure method to solve the utility maximization problem and obtained the associated results for a geometric Lévy process model.

In this paper we study the utility maximization problem for discrete-time incomplete markets using the method of martingale measure method. We first introduce the general results of utility maximization problem for a general discrete-time incomplete market. For a given utility function we construct a random variable following Karatzas, Lehoczky, Shreve and Xu (1991). If one of these variables can be replicated by a self-financing trading strategy, then the strategy is optimal and the associated martingale measure is also "optimal" in some sense. For the utility function $\log x$ (resp. $\frac{1}{\gamma}(x^\gamma - 1)(\gamma < 0)$), the associated martingale measure minimizes the relative entropy (resp. the Hellinger-Kakutani distance of order $\frac{\gamma}{\gamma-1}$); for utility function $-e^{-x}$(resp. $-(1-\gamma x)^{\frac{1}{\gamma}}(\gamma < 0)$), the associated martingale measure minimizes the relative entropy (resp. the Hellinger-Kakutani distance of order $\frac{\gamma}{\gamma-1}$) of dual form. The major characteristics of this paper is that for a special discrete-time market model, we work out explicitly the optimal trading strategies and the associated "optimal" equivalent martingale measures for the above two classes of utility functions.

APPENDIX

Proof of Lemma 4.1. Since for $a \in \left(-\frac{1+r_n}{u_n-r_n}, \frac{1+r_n}{r_n-d_n}\right)$ and $x \in [d_n, u_n]$ we have

$$(1+r_n+a(x-r_n))^{\gamma-1} \le (1+r_n-a(r_n-d_n))^{\gamma-1} \wedge (1+r_n+a(u_n-r_n))^{\gamma-1},$$

we see that $f_n(a)$ is well-defined. Let $a \in \left(-\frac{1+r_n}{u_n-r_n}, \frac{1+r_n}{r_n-d_n}\right)$, and $\varepsilon > 0$ such that $a+\varepsilon \in \left(-\frac{1+r_n}{u_n-r_n}, \frac{1+r_n}{r_n-d_n}\right)$. Since

$$(x-r_n)\left[(1+r_n+(a+\varepsilon)(x-r_n))^{\gamma-1} - (1+r_n+a(x-r_n))^{\gamma-1}\right] < 0,$$

for $x \in [d_n, u_n]$, it follows that

$$\begin{aligned}
& f_n(a+\varepsilon) - f_n(a) \\
= & \int_{\mathbb{R}} (x-r_n)\left[(1+r_n+(a+\varepsilon)(x-r_n))^{\gamma-1}\right. \\
- & \left.(1+r_n+a(x-r_n))^{\gamma-1}\right] F_n(dx) < 0.
\end{aligned}$$

Thus $f_n(a)$ is a strict decreasing function in $\left(-\frac{1+r_n}{u_n-r_n}, \frac{1+r_n}{r_n-d_n}\right)$. By the monotone convergence theorem, $f_n(a)$ is also continuous in $\left(-\frac{1+r_n}{u_n-r_n}, \frac{1+r_n}{r_n-d_n}\right)$. Then the conclusion follows. ∎

Proof of Theorem 4.2. From (12) we have

$$
\begin{aligned}
Z_{g^*}(n) &= \prod_{k=1}^{n} \frac{\left(1+\pi_k^* \frac{R_k - r_k}{1+r_k}\right)^{\gamma-1}}{\mathbb{E}\left[\left(1+\pi_k^* \frac{R_k - r_k}{1+r_k}\right)^{\gamma-1}\right]} \\
&= \prod_{k=1}^{n} \frac{(1+r_k+\pi_k^*(R_k-r_k))^{\gamma-1}}{\mathbb{E}\left[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma-1}\right]}.
\end{aligned}
$$

Since $\forall k = 1, \cdots, n,\ \pi_k^* \in \left(-\frac{1+r_k}{u_k - r_k}, \frac{1+r_k}{r_k - d_k}\right)$, we have $1+r_k+\pi_k^*(R_k-r_k) > 0$, thus $Z_{g^*}(n)$ is strictly positive. Obviously $(Z_{g^*}(n))$ is a martingale, thus $g^* \in \mathbb{G}$. On the other hand, since π_n^* is the solution of $f_n(a) = 0$, we have $f_n(\pi_n^*) = 0$, that is $\mathbb{E}[(R_n - r_n)(1+r_n+\pi_n^*(R_n-r_n))^{\gamma-1}] = 0$. Consequently (13) holds for g^*. Thus from Theorem 4.1 we know that $\mathbb{Q}_{g^*} \in \mathcal{P}$. Therefore, $(\widetilde{V}_n^{z,\pi^*})$ is a $\mathbb{Q}_{g^*}$-martingale, that is, $\mathbb{E}[Z_{g^*}(n)\beta_n V_n^{z,\pi^*}] = z$.

Recall that $V_n^{z,\pi^*} = z \prod_{k=1}^{n} [1+r_k+\pi_k^*(R_k-r_k)]$. Thus

$$
Z_{g^*}(n) V_n^{z,\pi^*} = z \prod_{k=1}^{n} \frac{(1+r_k+\pi_k^*(R_k-r_k))^{\gamma}}{\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma-1}]} .
$$

Therefore,

$$
\frac{1}{\beta_n} = \frac{1}{z}\mathbb{E}(Z_{g^*}(n) V_n^{z,\pi^*}) = \prod_{k=1}^{n} \frac{\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma}]}{\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma-1}]}.
$$

On the other hand, for $\delta = \frac{\gamma}{\gamma-1}$ and $n = 1, \cdots, N$, we have

$$
Z_{g^*}^{\delta}(n) = \prod_{k=1}^{n} \frac{(1+r_k+\pi_k^*(R_k-r_k))^{\gamma}}{(\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma-1}])^{\delta}} ,
$$

$$
\mathbb{E}[Z_{g^*}^{\delta}(n)] = \prod_{k=1}^{n} \frac{\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma}]}{(\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma-1}])^{\delta}} .
$$

Thus

$$\begin{aligned}
\xi_n^{g^*}(z) &= \frac{z}{\beta_n}\frac{(Z_{g^*}(n))^{\frac{1}{\gamma-1}}}{\mathbb{E}[Z_{g^*}^{\delta}(n)]} \\
&= \frac{z}{\beta_n}\prod_{k=1}^{n}\frac{1+r_k+\pi_k^*(R_k-r_k)}{(\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma-1}])^{\frac{1}{\gamma-1}}} \\
&\quad\times\prod_{k=1}^{n}\frac{\left(\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma-1}]\right)^{\delta}}{\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma}]} \\
&= \frac{z}{\beta_n}\prod_{k=1}^{n}\frac{(1+r_k+\pi_k^*(R_k-r_k))\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma-1}]}{\mathbb{E}[(1+r_k+\pi_k^*(R_k-r_k))^{\gamma}]} \\
&= \frac{z}{\beta_n}\cdot\prod_{k=1}^{n}(1+r_k+\pi_k^*(R_k-r_k))\cdot\beta_n \\
&= z\prod_{k=1}^{n}(1+r_k+\pi_k^*(R_k-r_k)) = V_n^{z,\pi^*}, \quad n=1,2,\cdots,N.
\end{aligned}$$

Specifically, $\xi_N^{g^*}(z) = V_N^{z,\pi^*}$. Thus the conclusion follows. ▮

Proof of Lemma 4.2. According to the assumption that $R_n > -1$, we know that $\frac{R_n-r_n}{1+r_n} > -1$. Since

$$\left|\left((x-r_n)e^{-a\frac{x-r_n}{1+r_n}}\right)I_{\{|\frac{x-r_n}{1+r_n}|\le 1\}}\right| \le e^{|a|}(1+r_n),$$

Thus

$$\int_{\mathbb{R}}\left|(x-r_n)e^{-a\frac{x-r_n}{1+r_n}}\right|I_{\{|\frac{x-r_n}{1+r_n}|\le 1\}}F_n(dx) < \infty.$$

On the other hand, it is clear that there exists some $b_n(-\infty \le b_n \le 0)$ such that $\int_{\mathbb{R}}(x-r_n)e^{-a\frac{x-r_n}{1+r_n}}I_{\{|\frac{x-r_n}{1+r_n}|>1\}}F_n(dx)$ is $+\infty$ for $a < b_n$ and finite for $a > b_n$. Thus the first property of $h_n(a)$ follows.

For any $a \in (b_n,\infty)$ and $\varepsilon > 0$ we have

$$(x-r_n)e^{-(a+\varepsilon)\frac{x-r_n}{1+r_n}} - (x-r_n)e^{-a\frac{x-r_n}{1+r_n}} < 0, \quad \forall x \in \mathbb{R}.$$

Consequently $h_n(a+\varepsilon)-h_n(a) < 0$, which means that $h_n(a)$ is strictly decrease in (b_n,∞). By monotone convergence theorem, h_n is also continuous in (b_n,∞). Since

$$\begin{aligned}\lim_{a\uparrow\infty} h_n(a) &= \lim_{a\uparrow\infty}\int_{\mathbb{R}}(x-r_n)e^{-a\frac{x-r_n}{1+r_n}}F_n(dx)\\ &= \lim_{a\uparrow\infty}\int_{\{x\geq r_n\}}(x-r_n)e^{-a\frac{x-r_n}{1+r_n}}F_n(dx)\\ &+ \lim_{a\uparrow\infty}\int_{\{x<r_n\}}(x-r_n)e^{-a\frac{x-r_n}{1+r_n}}F_n(dx)\\ &= 0-\infty=-\infty\end{aligned}$$

Thus $h_n(a)=0$ has a unique solution $\widehat{\pi}_n\in(b_n,\infty)$ if and only if $\lim_{a\downarrow b_n} h_n(a) > 0$. ▌

Proof of Theorem 4.3. Similar to Theorem 4.2, from (12) we have

$$Z_{\widehat{g}}(n)=\prod_{k=1}^n\frac{e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}}{\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]}.$$

Obviously, $(Z_{\widehat{g}}(n))$ is a strictly positive martingale, thus $\widehat{g}\in\mathbb{G}$. Note that $\widehat{\pi}_k\in(b_k,\infty), R_k\in[d_n,u_n], a.s.$, thus $Z_{\widehat{g}}(N)$ is $\mathbb{P}$-square-integrable. Since $\widehat{\pi}_n$ is the solution of $h_n(a)=0$, we have $h_n(\widehat{\pi}_n)=0$. That is,

$$\mathbb{E}\left[(R_n-r_n)e^{-\widehat{\pi}_n\frac{R_n-r_n}{1+r_n}}\right]=0.$$

Thus (13) holds for $\widehat{g}$. From Theorem 4.1 we know that $\mathbb{Q}_{\widehat{g}}$ is an equivalent martingale measure and $\mathbb{Q}_{\widehat{g}}\in\mathcal{P}'$. Then $(\widetilde{V}_n(\psi^n))$ is a $\mathbb{Q}_{\widehat{g}}$-martingale, that is, $\mathbb{E}_{\mathbb{Q}_{\widehat{g}}}[\beta_nV_n(\psi^n)]=\mathbb{E}[Z_{\widehat{g}}(n)\beta_nV_n(\psi^n)]=z$. Therefore,

$$\begin{aligned}\frac{z}{\beta_n} &= \mathbb{E}[Z_{\widehat{g}}(n)V_n(\psi^n)]=\mathbb{E}\left[Z_{\widehat{g}}(n)\left(\frac{z}{\beta_n}+\sum_{k=1}^n\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}\right)\right]\\ &= \mathbb{E}\left[\frac{z}{\beta_n}\cdot\prod_{k=1}^n\frac{e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}}{\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]}\right]+\mathbb{E}\left[Z_{\widehat{g}}(n)\left(\sum_{k=1}^n\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}\right)\right]\\ &= \frac{z}{\beta_n}+\mathbb{E}\left[Z_{\widehat{g}}(n)\left(\sum_{k=1}^n\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}\right)\right].\end{aligned}$$

Then we have

$$\mathbb{E}\left[Z_{\widehat{g}}(n)\left(\sum_{k=1}^n\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}\right)\right]=0.$$

Thus

$$
\begin{aligned}
&\mathbb{E}[Z_{\widehat{g}}(n)\log(Z_{\widehat{g}}(n))] + \log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\\
=\ & \mathbb{E}\left[Z_{\widehat{g}}(n)\log\left(\prod_{k=1}^{n}\frac{e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}}{\mathbb{E}[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}]}\right)\right] + \log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\\
=\ & \mathbb{E}\left[Z_{\widehat{g}}(n)\sum_{k=1}^{n}\left(-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k} - \log\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\right]\\
+\ & \log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\\
=\ & \mathbb{E}\left[-\left(Z_{\widehat{g}}(n)\sum_{k=1}^{n}\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}\right) - Z_{\widehat{g}}(n)\log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\right]\\
+\ & \log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\\
=\ & \mathbb{E}\left[-Z_{\widehat{g}}(n)\log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\right] + \log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\\
=\ & -\log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)Z_{\widehat{g}}(n) + \log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right) = 0.
\end{aligned}
$$

Thus for $n = 1, \cdots, N$,

$$
\begin{aligned}
\eta_n^{\widehat{g}}(z) &= \frac{z}{\beta_n} + \mathbb{E}[Z_{\widehat{g}}(n)\log Z_{\widehat{g}}(n)] - \log Z_{\widehat{g}}(n)\\
&= \frac{z}{\beta_n} + \mathbb{E}[Z_{\widehat{g}}(n)\log Z_{\widehat{g}}(n)] - \log\left(\prod_{k=1}^{n}\frac{e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}}{\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]}\right)\\
&= \frac{z}{\beta_n} + \mathbb{E}[Z_{\widehat{g}}(n)\log Z_{\widehat{g}}(n)] - \sum_{k=1}^{n}\left[-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k} - \log\left(\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\right]\\
&= \frac{z}{\beta_n} + \mathbb{E}[Z_{\widehat{g}}(n)\log Z_{\widehat{g}}(n)] + \sum_{k=1}^{n}\widehat{\pi}_k\frac{R_k-r_k}{1+r_k} + \log\left(\prod_{k=1}^{n}\mathbb{E}\left[e^{-\widehat{\pi}_k\frac{R_k-r_k}{1+r_k}}\right]\right)\\
&= \frac{z}{\beta_n} + \sum_{k=1}^{n}\widehat{\pi}_k\frac{R_k-r_k}{1+r_k} = V_n(\psi^n).
\end{aligned}
$$

Particularly, $\eta_N^{\widehat{g}}(z) = V_N(\psi^N)$. Since $\widetilde{V}_n(\psi^N) = z + \beta_N \sum_{k=1}^{n} \widehat{\pi}_k \frac{R_k - r_k}{1+r_k}$, we know that $\psi^N \in \widehat{\Psi}_s^z$. Thus the conclusion follows. ∎

REFERENCES

Bajieux, I. and R. Portait, 1995b, Pricing contingent claims in incomplete markets using the numeraire portfolio. *Working paper.* George Washington Univ. and E.S.S.E.C.

Bühlmann, H., F. Delbaen, P. Embrechts, and A. N. Shiryaev, essch transforms in discrete financial market. *Preprint.*

Chan, T., 1999, Pricing contingent claims on stocks driven by Lévy processes. *The Annals of Applied Probability* **9**(2), 504-528.

Davis H. A. M., 1997, Option pricing in incomplete markets. In:*Mathematics of Derivative Securities, Dempster, A. H. and Pliska, S. R. (eds.).* Publications of the Newton Institute, Cambridge Univ. Press, 216-226.

Elliott, R. J. and D. B. Madan, 1998, A discrete time equivalent martingale measure. *Mathematical Finance* **8**, 127-152.

Gerber, H. U. and E. S. W. Shiu, 1994, Option pricing by Esscher transform. *Transactions Society of Actuaries* **46**, 51-92.

He, S. W., J. J. Li, and J.M. Xia, 1999, Financial market of finite discrete-time model. *Advances in Mathematics (China)* **28**(1), 1-28.

Kar1 Karatzas, L., 1989, Optimization problem in the theory of continuous trading. *SIAM Journal on Control and Optimization* **27**, 1221-1259.

Karatzas, L., J. P. Lehoczky, and S. E. Shreve, 1987, Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM Journal on Control and Optimization* **25**, 1557-1586.

Karatzas, L., J. P. Lehoczky, S. E. Shreve, and G. L. Xu, 1991, Martingale and duality methods for utility maximization in incomplete markets. *SIAM Journal on Control and Optimization* **29**, 702-730.

Li, P. and J. A. Yan, 1999, The growth optimal portfolio in a discrete-time financial market. *Preprint.*

Long, J. B., 1990, The numeraire portfolio. *Journal of Financial Economics* **26**(1), 29-69.

Musiela, M. and M. Rutkowsky, 1997, *Martingale Methods in Financial Modelling: Theory and Applications*, Karatzas, I., Yor, M., eds, Berlin-Heidelberg-New York.

Schäl, M., 2000a, Portfolio optimization and martingale measures. *Mathematical Finance* **10**(2), 289-303.

Schäl, M., 2000b, Price systems constructed by optimal dynamic portfolios. *Mathematical Methods of Operations Research* **51**(3).

Xia, J. M. and J. A. Yan, 2000a, Martingale measure method for expected utility maximization and valuation in incomplete markets. *Preprint.*

Xia, J.M. and J.A. Yan, 2000b, The utility maximization approach to a martingale measure constructed via esscher transform. *Preprint.*

Yan, J. A., 1998, Introduction to maritngale methods in option pricing, *Lecture Notes in Mathematics, 4*, Liu Bie Ju Centre for Mathematical Sciences, City Univ. of Hong Kong.

Yan, J. A., 1998, *Lectures on Measure Theory*, Chinese Science Press.

Yan J. A., Q. Zhang, and S. G. Zhang, 2000, Growth optimal portfolio in a market driven by a jump-diffusion-like process or a lévy process. *Annals of Economics and Finance* **1**(1), 101-116.