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Term Structure of Interest Rates in the Presence of Lévy Jumps: The HJM Approach

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The aim of this paper is to generalize Heath, Jarrow and Morton (1992, Econometrica) model of the term structure of interest rates within a jump-diffusion formework. This is achieved by assuming that the forward rate process has a Lévy jump component with general jump size distributions. Sufficient conditions are derived under which the no-arbitrage condition implies the existence of a unique martingale measure within the jump-diffusion framework.

Key Words: Term structure of interest rates; Martingale measure; Jump-diffusion.
JEL Classification Numbers: G10, G12, G13.

1. INTRODUCTION

This paper studies no-arbitrage as a useful restriction on the term structure of interest rates and explores its implications on the risk neutral probability measure and the underlying short rate process for the purpose of bond pricing and pricing contingent claims on fixed income securities. It follows the approach of Heath, Jarrow and Morton (HJM,1992) and Ho and Lee (1986) by taking the forward rate process and/or the (whole) initial term structure of interest rates as given. First, constraints on the coefficients of the term structure movements that are consistent with the no-arbitrage conditions are constructed. Second, the set of martingale measures that are consistent with the no-arbitrage conditions is fully characterized. Third, the paper provides also conditions under which a unique martingale measure can be revealed from the bond prices. With the de-

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rived risk neutral martingale measure one can price any derivative product on fixed income securities. Therefore, a class of term structure of interest rates model emerges by incorporating Lévy jump process as a driving force in modelling bond price movements.

Following the fundamental theorem of asset pricing, á la Harrison and Kreps (1989), the absence of arbitrage implies the existence of a risk neutral probability measure $\mathbb{Q}$, which is absolutely continuous with respect to the objective probability measure $\mathbb{P}$ that governs the occurrence of the state of nature $\omega \in \Omega$, so that the time-t price C_t of a contingent claim X_T at maturity date T is given by

$$C_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} X_T \mid \mathbb{F}_t\right], \tag{1}$$

where $\mathbb{F}_t$ is the time-t information, and where r_t is the time-t instantaneous rate of return for the short term risk free bond. The time-t price $B(t,T)$ of the discount bond with maturity date T is thus determined by setting $X_T = 1$ in the above expression. In other words, determining the price of a contingent claim requires a knowledge of the risk-neutral probability measure $\mathbb{Q}$, particularly the probability distributions that govern the short rate process $\{r_t\}$ and the maturity payoff X_T under the $\mathbb{Q}$-measure.

The literature on pricing contingent claims, hence on determining the risk neutral probability measure $\mathbb{Q}$, can be roughly divided into two broad, but closely related, approaches: The general equilibrium approach and the no-arbitrage approach. The former is pioneered by Cox, Ingersoll and Ross (CIR, 1985a, b), and is further extended by Ahn and Thompson (1988) and Duffie and Epstein (1992), in which both the interest rate process $\{r_t\}_{t\geq 0}$ and the risk neutral probability measure $\mathbb{Q}$ are determined by the equilibrium conditions; that is, both the commodity and financial markets are cleared simultaneously given the pricing rule characterized by $\left(\mathbb{Q}, \{r_t\}_{t\geq 0}\right)$ for all tradable securities. The information structure in the general equilibrium framework involves not only the observations of the underlying security prices, but also the observations of some macro-economic variables such as the aggregate consumption and its growth rate. In particular, preference parameters and parameters for governing the movements of those macro-variables are all relevant in determining the risk neutral probability measure $\mathbb{Q}$ and the underlying short rate process $\{r_t\}_{t\geq 0}$. Therefore, estimating the preference parameters and other parameters associated with those macro-variables is an unavoidable task for the empirical implementation of the equilibrium approach.[1] Of course, it also involves the estimation of those parameters that govern the security price movements (such as

[1]CIR assume the log-utility function which makes its empirical implementation less challenging. For general utility consideration, please see Turnbull and Milne (1991) and

drifts and volatilities, etc). The pricing rule generated by the equilibrium approach will necessarily rule out all arbitrage opportunities. It is in this sense we say that, the no arbitrage condition is part of the equilibrium formulation.

Ho and Lee (1988), Vasicek (1977) and HJM (1992) are among the most well-known for developing the no-arbitrage models of term structure of interest rates. The no arbitrage approach involves the following common assumptions. First, the information structure is restricted to the current and historic prices of all relevant tradable securities. For example, Ho and Lee (1988) and HJM (1992) assume respectively the observations of the whole up-to-date term structure of interest rates $\{B(t,T)\}_{T\geq t\geq 0}$ and/or forward rates,[2] and Vasicek (1977) assumes the observation of long and short bond prices in a two-factor economy. Second, the no arbitrage condition is the only constraint imposed on all tradable securities. The objective of the no arbitrage approach is to "reveal", from the pre-specified information structure, the underlying risk-neutral probability measure $\mathbb{Q}$ and the short-rate process $\{r_t\}_{t\geq 0}$. Each of these papers provides some examples to illustrate how a "unique" risk neutral measure $\mathbb{Q}$ and $\{r_t\}_{t\geq 0}$ can be (partially) recovered from the pre-specified information structure by imposing purely the no arbitrage restrictions on the underlying security price movements. Nevertheless, all involve strong assumptions about the information structure and on the motion of the underlying term structure of interest rates.

So, generally speaking, in the presence of Lévy jumps, a unique risk neutral measure that is consistent with the no arbitrage conditions is difficult to derive since in contrast to the pure Brownian motion case studied by HJM (1992) and by Duffie and Kan (1994,96), the market is generally incomplete in the presence of Lévy jumps; and no arbitrage does not necessarily imply the existence of a unique risk-neutral measure (see, Ahn and Thompson 1988, Naik and Lee 1990 and Ma 1992 & 2000). In fact, a set of risk neutral probability measures could be identified, and each of which is consistent with the no arbitrage restrictions. This, nevertheless, does not necessarily diminish the usefulness of such an approach because the arbitrage conditions indeed constitute important restrictions on the motions of security prices, and what can be learned from such restrictions regarding the set of risk neutral measures, is obviously of concern to both economic theorists and practitioners.

Ma (1998) in discrete time, and Ahn and Thompson (1988) and Duffie and Epstein (1992) in continuous-time.

[2]Precisely, HJM assume the observation of the forward curve $\{f(t,T)\}_{T\geq t\geq 0}$, which is equivalent to the observation of the whole term structure of interest rates $\{B(t,T)\}_{T\geq t\geq 0}$.

Therefore, the objective of this paper is to establish links among the set of risk neutral measures, the forward rate process (hence short rate process) and/or the term structure of interest rates as consequences of no-arbitrage. More specifically, in the presence of Lévy jumps, we want to derive conditions on the dynamic of bond prices so that they are fully consistent with the existence of a risk-neutral measure even though such martingale measures may not be unique. Finally, sufficient conditions are provided under which a unique measure can be determined.[3]

The rest of the paper is organized as follows. Section 2 is a preliminary which contains a formal treatment of the information structure and the market structure for fixed income securities. The relationship between the term structure of interest rates and the underlying forward rate process is established. The notion of no-arbitrage is introduced in section 3, together with a full characterization of the term structure of interest rates and forward rate process that are consistent with the no-arbitrage restrictions. The dynamics of the forward rate, bond prices and short term interest rate under the risk neutral measure are respectively studied in section 4. Section 5 concerned with the uniqueness of risk neutral probability measure that is implied from the bond prices of all maturities. Section 6 contains some concluding remarks.

2. THE INFORMATION STRUCTURE

We assume as given a filtered probability space $\{\Omega, \mathbb{F}, \mathbb{P}\}$ with an increasing and right continuous filtration $\mathbb{F} = \{\mathbb{F}_t\}_{t \geqslant 0}$, and we start with a brief description of the Lévy jump process defined on $\{\Omega, \mathbb{F}, \mathbb{P}\}$:

A real valued process $\{x_t\}_{t \geq 0}$ is called $\mathcal{F}_t$-adapted if $x(t, \cdot)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$, and it is called $\mathcal{F}_t$- progressively measurable if $x(\cdot, \cdot) : [0, t] \times \Omega \to \mathcal{R}$ is $\mathcal{B}([0, t]) \times \mathcal{F}_t$-measurable for each $t \geq 0$, where $\mathcal{B}([0, t])$ is the Borel σ-algebra on $[0, t]$. A Lévy process is a $\mathcal{F}_t$- progressively measurable process that satisfies the following two properties: (a) its sample path $t \to x(t, \omega)$ is right continuous with finite left limit (i.e. RCLL), and (b) it has stationary and independent increments.

Given $\{N_t\}_{t \geq 0}$, an $\mathbb{F}_t$-adapted Lévy process, let $\Delta N_t \equiv N_t - N_{t-}$ be the size of a jump that occurs to N_t at time t. Let $\mathcal{B}(\mathcal{R})$ be the Borel σ-algebra

[3]Given the theoretical difficulty and mathematical complexity associated with the generalization, introducing the Lévy jump process into the model has also its well-documented (theoretical and empirical) advantages in modelling the security price moments (See Ahn and Thompson 1988, Brown and Dybvig 1986, Cox and Ross 1976, Naik and Lee 1990, and Ma 1992&2000) over models that are driven purely by Brownian motion. Particularly for fixed income securities, the jump has a natural appeal in modelling credit risk. For example, an upward jump in bond prices can be interpreted as a response to a higher hierarchical credit rating, and vise versa.

of the Euclidean space $\mathcal{R}$. For all $\Gamma \in \mathcal{B}(\mathcal{R}), \upsilon(t,\Gamma) \equiv \sum_{0<s\leq t} 1_{\Gamma}(\Delta N_s)$ is the number of jumps of sizes in Γ for $\{N_t\}$ that takes place in a time interval of length t. This defines a $\mathbb{F}_t$-adapted Poisson process $\{\upsilon(t,\Gamma)\}_{t\geq 0}$ with parameter $\mu(\Gamma) \equiv \mathbb{E}[\upsilon(1,\Gamma)] \geq 0$, where $1_{\Gamma}(u) = 1$ and 0 respectively for $u \in \Gamma$ and $u \notin \Gamma$. Denote by $\mathcal{N}$ the set of non-negative integers. The measures $\upsilon(\cdot,\cdot)$ and $\mu(\cdot)$ are respectively referred to as the *random Poisson measure* and the *Lévy measure* for the Lévy process $\{N_t\}_{t\geqslant 0}$. The random Poisson measure fully characterizes the probabilistic properties of the jumps associated with the Lévy process $\{N_t\}_{t\geq 0}$.

In the following, we restrict attention to nonexplosive Lévy process so that, with probability one $\sup_{\Gamma\in\mathcal{B}(\mathcal{R})}\{\upsilon(t,\Gamma)\} < \infty$ for all $t \geq 0$. Similarly, the Lévy measure $\mu(\cdot) : \mathcal{B}(\mathcal{R}) / \{0\} \to \mathcal{R}_+$, which determines the jump intensity for each of the counting processes $\{\upsilon(t,\Gamma)\}_{t\geq 0}$, can be regarded as a relative measure of the frequency of the jumps within different size categories. For example, for $\Gamma_1, \Gamma_2 \in \mathcal{B}(\mathcal{R}) / \{0\}, \Gamma_1 \cap \Gamma_2 = \emptyset$, the inequality $\mu(\Gamma_1) < \mu(\Gamma_2)$ can be interpreted as follows: for any given time period, more jumps with sizes in Γ_2 than those in Γ_1 will be expected.

2.1. The forward rates

The economy is assumed to contain two different sources of uncertainty. The first source of uncertainty comes from an n-dimensional standard Brownian motion $\{W_t\}_{t\geq 0}$ on $\{\Omega, \mathbb{F}, \mathbb{P}\}$ with continuous sample paths. The other source of uncertainty, which is assumed to be independent of the Brownian motions, comes from Lévy jumps $\{N_t\}_{t\geq 0}$. The jump process is totally characterized by the random Poisson measure $\{\upsilon(t,\cdot)\}_{t\geq 0}$ with parameter $\mu(\cdot)$.

Following HJM (1992), we assume that the motion of the state of the economy is fully characterized by the forward rate process $\{f(t,T)\}_{T\geq t\geq 0}$, $\forall T$. Therefore, $\mathbb{F}_t$ is specified to be the smallest σ-algebra that contains all forward curve at and before t; that is, for all $T \geq s$ and $s \leq t$, $f(s,T)$ is $\mathbb{F}_t$-measurable. The motion of the forward rate process $\{f(t,T)\}_{T\geq t\geq 0}$, for any given T, is assumed to follow a stochastic differential-difference equation (SDDE):

$$\begin{aligned} df(t,T) &= \alpha(t,T,f(t,T))\,dt + \sigma(t,T,f(t,T)) \cdot dW_t \\ &\quad + \int_{\mathcal{R}} \gamma(t,T,f(t,T),u)\,\upsilon(dt,du), \end{aligned} \tag{2}$$

with initial forward rate curve $\{f(0,T)\}_{T\geq 0}$ observed at $t = 0$.

In the above, α, σ and γ are all non-random continuously differentiable functions with respect to the first three arguments, and they satisfy also the so-called growth condition described below. These conditions are to ensure the existence and a unique solution to the SDDE (See Gihman and

Skorohod 1972, §3 Theorem 2 and §7 Theorem 1, Part II), noticing that the continuously differentiability assumptions are sufficient for the local-Lipschitz conditions.[4]

The Growth Conditions There exists a constant L such that, for all $T \geq t \geq 0, x \in \mathcal{R}$,

$$\begin{aligned} \alpha^2 (t, T, x) + & \parallel \sigma (t, T, x) \parallel^2 + \int_{\mathcal{R}} \gamma^2 (t, T, x, u) \mu (du) \\ \leq & L \left(1 + x^2\right), \text{ and} \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial \alpha (t, T, x)}{\partial t}\right)^2 + & \parallel \frac{\partial \sigma (t, T, x)}{\partial t} \parallel^2 + \int_{\mathcal{R}} \left(\frac{\partial \gamma (t, T, x, u)}{\partial t}\right)^2 \mu (du) \\ \leq & L(1 + x^2). \end{aligned}$$

where $\parallel \cdot \parallel$ is the Euclidean norm for $\mathcal{R}^n$.

Remark 2.1. In equation (2), when a jump of size u occurs at time t,i.e., $\triangle N_t = u$, the corresponding state variable jumps from f_{t-} to $f_{t+} = f_{t-} + \gamma_{t-} (\cdot, u)$. The subscript "$\pm$" is to emphasize that a jump occurs at time t, which is used throughout this paper for all random variables. Sometimes, we drop subscript "+" for notational simplicity.

Remark 2.2. The integral in (2), with respect to u, is understood as the stochastic Stieltjes integration. Loosely speaking, the integration aggregates the effects on f of all possible size of jumps that may occur in a small time interval $(t, t + dt]$.

For notational simplicity, we use $\alpha (t, T), \sigma (t, T)$ and $\gamma (t, T, u)$ to represent respectively for $\alpha (t, T, f (t, T)), \sigma (t, T, f (t, T))$ and $\gamma (t, T, f (t, T), u)$. We introduce also the following notions:

$$\begin{array}{lcl} \alpha^* (t, T) & \equiv & \int_t^T \alpha (t, s, f (t, s)) \, ds, \\ \sigma^* (t, T) & \equiv & \int_t^T \sigma (t, s, f (t, s)) \, ds, \\ \gamma^* (t, T, u) & \equiv & \int_t^T \gamma (t, s, f (t, s), u) \, ds. \end{array} \tag{3}$$

[4]Protter (1990) considers SDDE for general semi-martingales that contain the Lévy process described here as a special case. The corresponding SDDE for Lévy processes admit the same mathematical expression as above following the Lévy Decomposition Theorem (Protter 1990, Theorem 42 in Chapter 1), and from the definition of stochastic integration (Protter 1990, pp.50-51).

2.2. The term structure of interest rates

Given the observation of the forward rates, $\{f(t,T)\}_{T\geq t\geq 0}$,at time t, we are able to determine uniquely the price $B(t,T)$ of the discount bond with any maturity T, hence the time-t term structure of interest rates. We have[5], for all $0 \leq t \leq T$,

$$B(t,T) = \exp\left\{-\int_t^T f(t,s)\,ds\right\}, \text{ and } r_t = f(t,t), \tag{4}$$

where r_t is the risk-free interest rate for short term risk-free bond. Moreover, we can prove the following:

PROPOSITION 1. *Given (2) for the forward rates* $\{f(t,T)\}_{T\geq t\geq 0}$. *The discount bond price process* $B(\cdot,T)$ *must solve the SDDE,*

$$\begin{aligned}
\frac{dB(t,T)}{B(t,T)} &= b(t,T)\,dt + a(t,T)\cdot dW_t + \int_{\mathcal{R}} l(t,T,u)\,\upsilon(dt,du), \qquad (5)\\
B(T,T) &= 1, \forall 0 \leq t \leq T < \infty,
\end{aligned}$$

with coefficients given by

$$\begin{aligned}
b(t,T) &= r_t - \alpha^*(t,T) + \tfrac{1}{2} \parallel \sigma^*(t,T) \parallel^2,\\
a(t,T) &= -\sigma^*(t,T),\\
l(t,T,u) &= \exp(-\gamma^*(t,T,u)) - 1.
\end{aligned} \tag{6}$$

Proof. The bond price follows a jump-diffusion process with unknown, yet to be determined coefficients b, a and l. Applying Itô's Lemma to $\ln B(t,T)$, we have, for all $0 \leq t \leq T$,

$$\begin{aligned}
d\ln B(t,T) &= \left(b(t,T) - \frac{1}{2}a(t,T)\cdot a(t,T)\right)dt + a(t,T)\cdot dW_t\\
&\quad + \int_{\mathcal{R}} \ln(1 + l(t,T,u))\,\upsilon(dt,du).
\end{aligned}$$

[5]Such a relationship between bond price and forward rates is established by imposing the no arbitrage restrictions following any standard textbook. Therefore, the no arbitrage condition as part of the restriction on bond pricing is implicitly imposed in this formulation. The full-range implications of the no arbitrage restrictions on bond pricing remain to be explored, which is to be carried out in the next two sections.

Taking the partial derivative with respect to T to both sides of the equation, we have:

$$
\begin{aligned}
d\frac{\partial}{\partial T}\ln B(t,T) &= \left(\frac{\partial b(t,T)}{\partial T} - a(t,T)\cdot\frac{\partial a(t,T)}{\partial T}\right)dt + \frac{\partial a(t,T)}{\partial T}\cdot dW_t \\
&\quad + \int_{\mathcal{R}} \frac{1}{1+l(t,T,u)}\frac{\partial l(t,T,u)}{\partial T}\upsilon(dt,du).
\end{aligned}
$$

From (4), we see that

$$
f(t,T) = -\frac{\partial}{\partial T}(\ln B(t,T)), \forall 0 \le t \le T.
$$

This, together with (2), leads to the following ordinary differential equations for the coefficients:

$$
\begin{aligned}
\frac{\partial b(t,T)}{\partial T} &= -\alpha(t,T) + a(t,T)\cdot\frac{\partial a(t,T)}{\partial T}, \\
\frac{\partial a(t,T)}{\partial T} &= -\sigma(t,T), \\
\frac{1}{1+l(t,T,u)}\frac{\partial l(t,T,u)}{\partial T} &= \gamma(t,T,u),
\end{aligned} \tag{7}
$$

with initial conditions: $b(t,t) = r_t$, $a(t,t) = 0, l(t,t,u) = 0$ at $T = t$. The solutions to these equations are thus given by (6). ▮

3. ABSENCE OF ARBITRAGE AND BOND PRICES

We consider a market that contains a continuum number of securities by allowing discount bonds of all maturities, as well as their derivative products, to be available for trade. A portfolio at any point of time may involve a holding of an arbitrary, but finite, number of securities with different maturities.[6] Absence of arbitrage for the whole market will necessarily imply the absence of arbitrage with respect to trading of any arbitrarily fixed number of securities. Therefore, we adopt the following as the definition of no-arbitrage in our framework:

Definition 3.1. The market is said to admit no arbitrage if there is a probability measure $\mathbb{Q}$, that is equivalent to $\mathbb{P}$, such that, for any given contingent claim with maturity payoff X_T at T that is $\mathbb{F}_T$-measurable, its time-t price C_t is given by

$$
C_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} X_T \mid \mathbb{F}_t\right], \forall t \le T < \infty. \tag{8}
$$

[6]Theoretically, we may consider portfolios that contains any arbitrary number of securities as well.

The measure $\mathbb{Q}$ is referred to as risk neutral measure, or a martingale measure.[7]

In this definition, the risk-neutral measure $\mathbb{Q}$ is common to all tradable securities, and is invariant to the time at maturity T. The risk-neutral pricing rule such defined admits no arbitrage opportunities for any given fixed number of securities in the sense of Harrison and Kreps (1989). It holds particularly true for the bond market that contains discount bond of all maturities.

Here is the main result of this section:

THEOREM 1. *If there exists no arbitrage, then there exist* $\psi : \mathcal{R}_+ \times \Omega \to \mathcal{R}^n$ *and* $\varphi : \mathcal{R}_+ \times \mathcal{R} \times \Omega \to \mathcal{R}_{++}$*, which are continuously differentiable with respect to* t *and* $\mathbb{F}_t$*- progressively measurable, and satisfy*[8]

$$
\begin{aligned}
\int_{\mathcal{R}} \varphi(t,u)\,\mu(du) &< \infty, \mathbb{P}\text{-}a.s., \\
\int_{\mathcal{R}} e^{-\gamma^*(t,T,u)} \varphi(t,u)\,\mu(du) &< \infty, \mathbb{P}\text{-}a.s., \\
\int_{\mathcal{R}} \gamma(t,T,u)\, e^{-\gamma^*(t,T,u)} \varphi(t,u)\,\mu(du) &< \infty, \mathbb{P}\text{-}a.s.,
\end{aligned}
$$

such that

$$
\begin{aligned}
\alpha(t,T) &\equiv \sigma(t,T) \cdot [\sigma^*(t,T) - \psi(t)] - \\
&\quad \int_{\mathcal{R}} \gamma(t,T,u)\, e^{-\gamma^*(t,T,u)} \varphi(t,u)\,\mu(du), \qquad (9) \\
b(t,T) &\equiv r_t + \sigma^*(t,T) \cdot \psi(t) + \\
&\quad \int_{\mathcal{R}} \left(1 - e^{-\gamma^*(t,T,u)}\right) \varphi(t,u)\,\mu(du). \qquad (10)
\end{aligned}
$$

Conversely, if such ψ *and* φ *exist, then the bond market that contains discount bond of all maturities admits no arbitrage.*

The prove the theorem, we need to make use of the following Lemmas:

[7] A probability measure $\mathbb{Q}$ on $(\Omega, \mathbb{F})$ is said to be *equivalent* to $\mathbb{P}$ if, for all $E \in \mathbb{F}$, $\mathbb{Q}(E) > 0 \Leftrightarrow \mathbb{P}(E) > 0$. A measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ is called a *martingale measure* if the present value of any tradable security follows a martingale under $\mathbb{Q}$.

[8] The differentiability conditions can be relaxed. The argument $\omega \in \Omega$ is dropped for notational simplicity.

Lemma 1. *Let $g(t)$ and $h(t,u) > -1$ be respectively $\mathbb{F}_t$- and $\mathbb{F}_t \times \mathcal{B}(\mathcal{R})$-progressively measurable processes. Suppose that*

$$\mathbb{E}\left[\int_0^t \| g(s) \|^2 \, ds\right] < \infty, \tag{11}$$

$$\mathbb{E}\left[\int_0^t \int_{\mathcal{R}} h(s,u)\, \mu(du)\, ds\right] < \infty, \tag{12}$$

$$\mathbb{E}\left[\int_0^t \int_{\mathcal{R}} \ln(1 + h(s,u))\, \mu(du)\, ds\right] < \infty. \tag{13}$$

Define, for all $t \geq 0$,

$$\begin{aligned}\zeta_t &= \exp\{\int_0^t g(s) \cdot dW_s - \frac{1}{2}\int_0^t \| g(s) \|^2 \, ds \\ &+ \int_0^t \int_{\mathcal{R}} \ln(1 + h(s,u))\, \upsilon(ds,du) - \int_0^t \int_{\mathcal{R}} h(s,u)\, \mu(du)\, ds\}. \end{aligned} \tag{14}$$

We have:

(a)

$$\frac{d\zeta_t}{\zeta_t} = -\int_{\mathcal{R}} h(t,u)\, \mu(du)\, dt + g(t) \cdot dW_t + \int_{\mathcal{R}} h(t,u)\, \upsilon(dt,du), \tag{15}$$

with $\zeta_0 = 1$;

(b)$\{\zeta_t\}_{t \geq 0}$ is a positive $\mathbb{P}$-martingale: $\mathbb{E}[\zeta_s \mid \mathbb{F}_t] = \zeta_t > 0$ for all $s \geq t$.

Proof. Statement (a) follows by applying Itô's lemma to the process ζ as defined above. In particular, the underlying process satisfies also the following stochastic integral equation:

$$\zeta_s - \zeta_t = \int_t^s \zeta_\tau g(\tau) \cdot dW_\tau + \int_t^s \int_{\mathcal{R}} \zeta_\tau h(\tau,u)\, \widetilde{\upsilon}(d\tau,du), s \geq t,$$

where $\widetilde{\upsilon}(dt,du) \equiv \upsilon(dt,du) - \mu(du)\, dt$ is the normalized random Poisson measure.

To prove statement (b), first, set $t = 0$ and take the unconditional expectation on both hand sides of the above expression, noticing that ζ is a positive process, we have:

$$\mathbb{E}[| \zeta_s |] = \mathbb{E}[\zeta_s] = \mathbb{E}[\zeta_0] \equiv 1 < \infty, \forall s \geq 0.$$

Second, for all $s \geq t \geq 0$, taking the expectation on both hand sides of the equation conditional on time-t information $\mathbb{F}_t$, it yields

$$\mathbb{E}\left[\zeta_s \mid \mathbb{F}_t\right] - \zeta_t = 0,$$

since, by definition, ζ is $\mathbb{F}_t$-measurable.

Therefore, $\{\zeta_t\}_{t\geq 0}$ is a positive $\mathbb{P}$-martingale following Karatzas and Shreve (1988, Definition 3.1). ∎

LEMMA 2. *Let $\mathbb{Q}$ be a probability measure that is equivalent to $\mathbb{P}$, then*

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathbb{F}_t} = \zeta_t, \mathbb{P}\text{-}a.s., \tag{16}$$

where $\{\zeta_t\}_{t\geq 0}$, which is referred to as Radon-Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, is a positive $\mathbb{P}$-martingale as in Lemma 1 for some g and h for which $\mathbb{E}\left[\zeta_t\right] = 1, t \geq 0$.

Proof. See Jacod and Shiryaev (1987, Chapter III, Theorems 3.24 and 5.19) and Chan (1999, Theorem 3.2). ∎

LEMMA 3. *Let $\mathbb{Q}$ be as defined in Lemma 2. A $\mathbb{F}_t$-adapted process $\{X_t\}$ is a $\mathbb{Q}$-martingale if, and only if, $\{\zeta_t X_t\}$ is a $\mathbb{P}$-martingale.*

Proof. The statement follows by noticing that

$$\mathbb{E}^{\mathbb{Q}}\left[X_T \mid \mathbb{F}_t\right] = \frac{1}{\zeta_t}\mathbb{E}\left[\zeta_T X_T \mid \mathbb{F}_t\right], \forall 0 \leq t \leq T < \infty. \tag{17}$$

∎

Proof (**Proof of Theorem 1).** For the first part of the theorem, suppose there exists a measure $\mathbb{Q}$, that is equivalent to $\mathbb{P}$, such that for all $T < \infty$, the present value of the discount bond price, $\left\{\exp\left\{-\int_0^t r_s ds\right\} B(t,T)\right\}_{0\leq t\leq T}$ is a $\mathbb{Q}$-martingale. Let g, h and ζ be as in Lemmas 1 and 2, and let

$$X(t,T) \equiv \exp\left\{-\int_0^t r_s ds\right\} B(t,T), \forall t \leq T. \tag{18}$$

By Itô's Lemma, $X(t,T)$ satisfies the SDDE:

$$\frac{dX(t,T)}{X(t,T)} = \left[b(t,T) - r_t\right] dt + a(t,T)\cdot dW_t + \int_{\mathcal{R}} l(t,T,u)\, \upsilon(dt,du), \tag{19}$$

with an explicit solution given by

$$\begin{aligned} X(t,T) &= B(0,T)\exp\{\int_0^t \left[b(s,T) - r_s - .5 \parallel a(s,T) \parallel^2\right] ds \\ &+ \int_0^t a(s,T)\cdot dW_s + \int_0^t \int_{\mathcal{R}} \ln(1 + l(s,T,u))\, \upsilon(ds,du)\}. \end{aligned} \quad (20)$$

Consider

$$\begin{aligned} \zeta_t X(t,T) &= B(0,T)\exp\{\int_0^t \left[b(s,T) - r_s - .5\left(\parallel a(s,T)\parallel^2 + \parallel g(s)\parallel^2\right)\right] ds \\ &+ \int_0^t [a(s,T) + g(s)]\cdot dW_s - \int_0^t \int_{\mathcal{R}} h(s,u)\,\mu(du)\,ds \\ &+ \int_0^t \int_{\mathcal{R}} \ln[(1 + l(s,T,u))(1 + h(s,u))]\upsilon(ds,du). \end{aligned} \quad (21)$$

By assumption, and by Lemma 3, $\zeta_t X(t,T)$ is a $\mathbb{P}$-martingale. This, by Lemma 2, leads to the following equality:

$$\begin{aligned} & b(t,T) - r_t - .5\left(\parallel a(t,T)\parallel^2 + \parallel g(t)\parallel^2\right) - \int_{\mathcal{R}} h(t,u)\,\mu(du) \\ &= -.5 \parallel a(t,T) + g(t)\parallel^2 \\ &- \int_{\mathcal{R}} [(1 + l(t,T,u))(1 + h(t,u)) - 1]\mu(du), \end{aligned} \quad (22)$$

or, equivalently,

$$b(t,T) = r_t - a(t,T)\cdot g(t) - \int_{\mathcal{R}} l(t,T,u)(1 + h(t,u))\,\mu(du). \quad (23)$$

This reduces to equation (10) by Proposition 1 and by setting $\psi = g$ and $\varphi = 1+h$. The expression (9) for α follows from Proposition 1 with equation (10).

For the second part of the theorem, suppose α and b are expressed respectively as (9) and (10). Let ζ_t be as defined in Lemma 1 with $g = \psi$ and $h = \varphi - 1$,and let probability measure $\mathbb{Q}$ be such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathbb{F}_t} = \zeta_t, \forall t \geq 0. \quad (24)$$

Reverting the above procedure, we see that,

$$\begin{aligned}
\zeta_t X(t,T) &= B(0,T)\exp\{-.5\int_0^t \parallel a(s,T) \\
&+ \psi(s) \parallel^2 ds + \int_0^t [a(s,T)+\psi(s)]\cdot dW_s \\
&- \int_0^t \int_{\mathcal{R}} [(1+l(t,T,u))\,\varphi(t,u)-1]\mu(du)\,ds \\
&+ \int_0^t \int_{\mathcal{R}} \ln[(1+l(s,T,u))\,\varphi(s,u)]\upsilon(ds,du)\}. \qquad (25)
\end{aligned}$$

By Lemma 1, $\{\zeta_t X(t,T)\}_{0\le t\le T}$ is a $\mathbb{P}$-martingale. This implies, by Lemma 3, that $\{e^{-\int_0^t r_s ds}B(t,T)\}_{0\le t\le T}$ is a $\mathbb{Q}$-martingale.

These end the proof. ∎

Remark 3.1. Given b and α as respectively expressed by (9) and (10) for some ψ and φ, there may exist many pairs (ψ',φ'), each of which would make the expressions to be valid. If this is the case, the martingale measure that constructed above will not be unique. Conditions for a unique bundle (ψ,φ) for b and α,hence for the existence of a unique risk neutral measure, is to be explored in section 5 below.

4. BOND PRICES AND FORWARD RATES UNDER MEASURE $\mathbb{Q}$

This section is to study the dynamics of bond prices, forward rates and the interest rates under the martingale probability measure $\mathbb{Q}$. In addition to the previous assumptions (those assumed for Theorem 1), we restrict φ to be deterministic.

We consider the martingale measure $\mathbb{Q}$ corresponding to $(g,h)=(\psi,\varphi-1)$. That is $\frac{d\mathbb{Q}}{d\mathbb{P}}_{|\mathbb{F}_t}=\zeta_t$, with

$$\begin{aligned}
\zeta_t &= \exp\{\int_0^t \psi(s)\cdot dW_s - \frac{1}{2}\int_0^t \parallel \psi(s) \parallel^2 ds \\
&+ \int_0^t \int_{\mathcal{R}} \ln(\varphi(s,u))\,\upsilon(ds,du) \\
&+ \int_0^t \int_{\mathcal{R}} (1-\varphi(s,u))\,\mu(du)\,ds\}. \qquad (26)
\end{aligned}$$

The motion of bond prices under the original probability measure $\mathbb{P}$ is fully characterized by Theorem 1. To determine the motion of bond prices under the $\mathbb{Q}$ measure, we need to establish the following lemmas.

LEMMA 4. *The process* $W_t^* \equiv W_t - \int_0^t \psi(s)\,ds, t \geq 0$, *so defined, is a* m*-dimensional standard* $\mathbb{Q}$*-Brownian motion.*

Proof. We need to compute the characteristic function for W_t^* under the $\mathbb{Q}$-measure:

Given $\lambda \in \mathcal{R}^m$ and $t \geq 0$, we have,

$$
\begin{aligned}
& \mathbb{E}_0^{\mathbb{Q}}\left[e^{\lambda \cdot W_t^*}\right] \\
= \; & \mathbb{E}_0[\exp\{\lambda \cdot W_t^* + \int_0^t \psi(s) \cdot dW_s - .5\int_0^t \| \psi(s) \|^2 \, ds \\
& + \int_0^t \int_{\mathcal{R}} \ln(\varphi(s,u))\, \upsilon(ds,du) + \int_0^t \int_{\mathcal{R}} [1 - \varphi(s,u)]\, \mu(du)\, ds\}] \\
= \; & \mathbb{E}_0[\exp\{\| \lambda \|^2 \, t + \int_0^t (\psi(s) + \lambda) \cdot dW_s - .5\int_0^t \| \psi(s) + \lambda \|^2 \, ds \\
& + \int_0^t \int_{\mathcal{R}} \ln(\varphi(s,u))\, \upsilon(ds,du) + \int_0^t \int_{\mathcal{R}} [1 - \varphi(s,u)]\, \mu(du)\, ds\}] \\
= \; & \mathbb{E}_0\left[e^{\|\lambda\|^2 t} \zeta_t'\right] = e^{\|\lambda\|^2 t},
\end{aligned}
$$

where ζ_t' so defined is, by Lemma 1, a positive $\mathbb{P}$-martingale with $\mathbb{E}[\zeta_t' \mid \mathbb{F}_0] = 1, \forall t \geq 0$. The derived characteristic function is for m independent Normal random variables with zero means and standard deviations given by $\sqrt{t}$.

Moreover, for all $t = t_n > \cdots > t_0 = 0$, $\left\{W_{t_j}^* - W_{t_j - 1}^*\right\}_{1 \leq j \leq n}$ are independent random vectors since

$$
\begin{aligned}
\mathbb{E}_0^{\mathbb{Q}}\left[e^{\lambda \cdot \sum_{j=1}^n (W_{t_j}^* - W_{t_j-1}^*)}\right] &= e^{\|\lambda\|^2 t} \equiv \prod_{j=1}^n e^{\|\lambda\|^2 (t_j - t_{j-1})} \\
&= \prod_{j=1}^n \mathbb{E}_0^{\mathbb{Q}}\left[e^{\lambda \cdot (W_{t_j}^* - W_{t_j-1}^*)}\right].
\end{aligned}
$$

Therefore, W^* is a m-dimensional standard Brownian motion under the $\mathbb{Q}$-measure. ▮

LEMMA 5. *The Poisson process* $\{\upsilon((0,t],A)\}_{t\geq 0}, A \in \mathcal{B}(\mathcal{R})$, *under* $\mathbb{Q}$, *is non-stationary and has a Poisson measure* $\mu((0,t],A)$ *given by*[9]

$$\mu((0,t],A) \equiv \int_0^t \int_A \varphi(s,u)\,\mu(du)\,ds. \tag{27}$$

In particular, for all disjoint $\{A_j\}_{j=1}^J \subseteq \mathcal{B}(\mathcal{R})$, *and* $t = t_I > \cdots > t_0 = 0$, $\{\upsilon((t_{i-1},t_i],A_j)\}_{\substack{1 \leq i \leq I \\ 0 \leq j \leq J}}$ *are independent to each other, and are independent to the Brownian motion* W^* *defined above.*

Proof. We start with a computation of the characteristic function for $\upsilon((0,t],A)$ under measure $\mathbb{Q}$: For any given $\lambda \in \mathcal{R}$, we have

$$\begin{aligned}
&\mathbb{E}_0^{\mathbb{Q}}\left[e^{\lambda \upsilon((0,t],A)}\right] \\
=\ &\mathbb{E}_0[\exp\{\int_0^t \psi(s)\cdot dW_s - .5\int_0^t \parallel \psi(s) \parallel^2 ds \\
&+ \int_0^t \int_{\mathcal{R}} \ln\left[\varphi(s,u)\,e^{\lambda 1_A(u)}\right] \upsilon(ds,du) \\
&+ \int_0^t \int_{\mathcal{R}} [1-\varphi(s,u)]\,\mu(du)\,ds\}] \\
=\ &\mathbb{E}_0[\exp\{\int_0^t \psi(s)\cdot dW_s - .5\int_0^t \parallel \psi(s) \parallel^2 ds \\
&+ \int_0^t \int_{\mathcal{R}} \ln\left[\varphi(s,u)\,e^{\lambda 1_A(u)}\right] \upsilon(ds,du) \\
&+ \int_0^t \int_{\mathcal{R}} \left[1-\varphi(s,u)\,e^{\lambda 1_A(u)}\right] \mu(du)\,ds \\
&+ \int_0^t \int_{\mathcal{R}} \varphi(s,u)\left(e^{\lambda 1_A(u)}-1\right) \mu(du)\,ds\}] \\
=\ &\mathbb{E}_0\left[\zeta_t''\exp\left\{\left(e^{\lambda}-1\right)\int_0^t \int_A \varphi(s,u)\,\mu(du)\,ds\right\}\right] \\
=\ &\exp\left\{\left(e^{\lambda}-1\right)\mu((0,t],A)\right\},
\end{aligned}$$

where the process ζ'' so defined is a positive $\mathbb{P}$-martingale with $\mathbb{E}[\zeta_t'' \mid \mathbb{F}_0] = 1, \forall t \geq 0$. The right hand side of the last equation is the characteristic function for a Poisson random variable with jump intensity $\mu((0,t],A)$ as defined above.

[9]That is, $\upsilon(dt,du)$ has a instantaneous jump intensity given by $\varphi(t,u)\,\mu(du)\,dt$.

The first part of the second statement holds true because

$$\begin{aligned}
& \mathbb{E}_0^{\mathbb{Q}}\left[e^{\lambda \sum_{i,j} \upsilon((t_{i-1},t_i],A_j)}\right] \\
= \; & \exp\left\{\left(e^{\lambda}-1\right)\mu\left((0,t],\cup_j A_j\right)\right\} \\
= \; & \exp\left\{\left(e^{\lambda}-1\right)\sum_i\sum_j \mu\left((t_{i-1},t_i],A_j\right)\right\} \\
= \; & \prod_{i,j}\exp\left\{\left(e^{\lambda}-1\right)\mu\left((t_{i-1},t_i],A_j\right)\right\} \\
= \; & \prod_{i,j}\mathbb{E}_0^{\mathbb{Q}}\left[e^{\lambda\upsilon((t_{i-1},t_i],A_j)}\right].
\end{aligned}$$

Following the same procedure as above, we can verify that υ and W^* are independent; that is: for all t and $\tau \geq 0$, and for all λ and λ',

$$\mathbb{E}_0^{\mathbb{Q}}\left[e^{\lambda\upsilon((0,t],A)+\lambda'\cdot W_\tau^*}\right] = \mathbb{E}_0^{\mathbb{Q}}\left[e^{\lambda\upsilon((0,t],A)}\right] \times \mathbb{E}_0^{\mathbb{Q}}\left[e^{\lambda'\cdot W_\tau^*}\right].$$

The details are thus omitted. ▮

With these two lemmas, the following proposition follows immediately from Proposition 1 and Theorem 1:

Proposition 2. *Under measure $\mathbb{Q}$, the bond price process $\{B(t,T)\}_{0\leq t\leq T}$ must satisfy the following SDDE:*

$$\begin{aligned}
\frac{dB(t,T)}{B(t,T)} &= \left(r_t + \int_{\mathcal{R}}\left(1-e^{-\gamma^*(t,T,u)}\right)\varphi(t,u)\,\mu(du)\right)dt \\
&\quad -\sigma^*(t,T)\cdot dW_t^* - \int_{\mathcal{R}}\left(1-e^{-\gamma^*(t,T,u)}\right)\upsilon(dt,du) \\
&= r_t dt - \sigma^*(t,T)\cdot dW_t^* + \int_{\mathcal{R}}\left(e^{-\gamma^*(t,T,u)}-1\right)\widetilde{\upsilon}^*(dt,du), \quad (28)
\end{aligned}$$

where

$$\widetilde{\upsilon}^*(dt,du) \equiv \upsilon(dt,du) - \varphi(t,u)\,\mu(du)\,dt$$

is the normalized random Poisson measure under $\mathbb{Q}$. Similarly, under $\mathbb{Q}$, the forward rate process $\{f(t,T)\}_{0\leq t\leq T}$ follows

$$df(t,T) = \alpha'(t,T)\,dt + \sigma(t,T)\cdot dW_t^* + \int_{\mathcal{R}}\gamma(t,T,u)\,\upsilon(dt,du), \qquad (29)$$

where

$$\alpha'(t,T) \equiv \sigma(t,T)\cdot\sigma^*(t,T) - \int_{\mathcal{R}}\gamma(t,T,u)\,e^{-\gamma^*(t,T,u)}\varphi(t,u)\,\mu(du). \quad (30)$$

Remark 4.1. This result reduces to HJM (1992) when the forward rates are purely driven by the Brownian motions. Similar to the Brownian information case, under the risk neutral measure $\mathbb{Q}$, the discount bonds of all maturities share a common drift given by the risk-free interest rate r_t with the volatilities associated with the Brownian motions W^* and jump sizes associated with the random Poisson measure $\upsilon(\cdot,\cdot)$ respectively remaining the same as under the original measure $\mathbb{P}$. Nevertheless, the jump intensity, given by $\varphi(t,u)\,\mu(du)\,dt$, for the random Poisson measure under the risk neutral measure $\mathbb{Q}$, is different to that under the original measure $\mathbb{P}$ unless $\varphi \equiv 1$. In other words, the change of probability measure changes not only the drift of the bond prices but also the underlying jump intensities. Accordingly, the "cost of risk" interpretation of the difference between the drift of the bond prices, under the original measure, and the risk free interest rate is no longer valid in the presence of Lévy jumps except when $\varphi \equiv 1$.

It remains to derive the motion for the short term interest rate process $\{r_t\}$ under measure $\mathbb{Q}$. We have:

Proposition 3. *Under measure $\mathbb{Q}$, the interest rate process satisfies the following SDDE:*

$$dr_t = \eta_t dt + \sigma(t,t,r(t)) \cdot dW_t^* + \int_{\mathcal{R}} \gamma(t,t,r(t),u)\,\widetilde{\upsilon}^*(dt,du)\,, \tag{31}$$

where

$$\begin{aligned}\eta_t \;\equiv\; & \frac{\partial f}{\partial T}(0,t) + \int_0^t \frac{\partial \alpha'}{\partial T}(s,t)\,ds \\ & + \int_0^t \frac{\partial \sigma}{\partial T}(s,t)\cdot dW_s^* + \int_0^t \int_{\mathcal{R}} \frac{\partial \gamma}{\partial T}(s,t,u)\,\upsilon(ds,du)\,.\end{aligned} \tag{32}$$

Proof. First, by definition, we have:

$$\begin{aligned} r_t &= f(t,t) \\ &= f(0,t) + \int_0^t \alpha'(s,t)\,ds + \int_0^t \sigma(s,t)\cdot dW_s^* \\ &\quad + \int_0^t \int_{\mathcal{R}} \gamma(s,t,u)\,\upsilon(ds,du)\,. \end{aligned} \tag{33}$$

Applying the integration by parts and the stochastic Fubini theorem to each of the above terms, we have:

$$\begin{aligned}
& f(0,t) \\
= \;& r_0 + \int_0^t \frac{\partial f}{\partial T}(0,s)\,ds, \\
& \int_0^t \alpha'(s,t)\,ds \\
= \;& \int_0^t \alpha'(s,s)\,ds + \int_0^t \int_0^\tau \frac{\partial \alpha'}{\partial T}(s,\tau)\,ds d\tau \\
= \;& -\int_0^t \int_{\mathcal{R}} \gamma(s,s,u)\,\varphi(s,u)\,\mu(du)\,ds + \int_0^t \int_0^s \frac{\partial \alpha'}{\partial T}(\tau,s)\,d\tau ds, \\
& \int_0^t \sigma(s,t)\cdot dW_s^* \\
= \;& \int_0^t \sigma(s,s)\cdot dW_s^* + \int_0^t \int_0^s \frac{\partial \sigma}{\partial T}(\tau,s)\cdot dW_\tau^* ds, \\
& \int_0^t \int_{\mathcal{R}} \gamma(s,t,u)\,\upsilon(ds,du) \\
= \;& \int_0^t \int_{\mathcal{R}} \gamma(s,s,u)\,\upsilon(ds,du) + \int_0^t \left(\int_0^s \int_{\mathcal{R}} \frac{\partial \gamma}{\partial T}(\tau,s,u)\,\upsilon(d\tau,du) \right) ds.
\end{aligned}$$

Combining these expressions, with η as defined above, we have

$$r_t = r_0 + \int_0^t \eta_s ds + \int_0^t \sigma(s,s)\cdot dW_s^* + \int_0^t \int_{\mathcal{R}} \gamma(s,s,u)\,\widetilde{\upsilon}^*(ds,du), \quad (34)$$

which satisfies the SDDE described above since, by definition,

$$\sigma(s,s) \equiv \sigma(s,s,r(s)) \text{ and } \gamma(s,s,u) \equiv \gamma(s,s,r(s),u).$$

∎

Remark 4.2. This result generalizes Musiela and Rutkowski (1997, Proposition 13.1.1) by incorporating the Lévy jumps into the underlying short rate process. The short rate process is Markovian if η_t defined by equation (32) could be expressed as a function of (t, r_t).

5. ON THE UNIQUENESS OF $\mathbb{Q}$-MEASURE

This section is on the uniqueness of the martingale measure $\mathbb{Q}$. We ask the following question: Suppose, we know all the coefficients for the forward rate process and bond prices, which are fully consistent with no-arbitrage restrictions, can we determine a unique martingale measure so that we can price all interest rate sensitive contingent claims?

Following the remark after the proof of Theorem 1, we see that:

PROPOSITION 4. *Given α and b as expressed respectively by (9) and (10), no-arbitrage for the bond market implies the existence of a unique martingale probability measure $\mathbb{Q}$ if, and only if, $(g,h) = (\psi, \varphi - 1)$ constitute the unique solution to the following equation(s):*

$$\sigma^* (t,T) \cdot (g(t) - \psi(t)) + \int_{\mathcal{R}} \left(1 - e^{-\gamma^*(t,T,u)}\right) \Delta h(t,u)\, \mu(du) = 0, \mathbb{P}\text{-}a.s., \tag{35}$$

for all $T \geq t \geq 0$, where $\Delta h \equiv h - \varphi + 1$. In particular, any solution (g,h) to the above equation can be used to define ζ and the corresponding martingale measure $\mathbb{Q}$.

5.1. The case with finite number of jump sizes

Consider first the special case with jump sizes taking a finite number of possible values. That is,

A1. Finite discrete jump sizes:

$$\mu(du) = \sum_{k=1}^{m} \pi_k \delta[u - u_k], \pi_k > 0, \forall k. \tag{36}$$

The following version of full-rank condition is assumed:

A2. For all $t \geq 0$, there exists $T_1, \cdots, T_{n+m} \geq t$, such that the matrix $[\Sigma(t), L(t)]$ is non-singular, where

$$\begin{aligned} \Sigma_{ij}(t) &= \sigma_j^*(t,T_i), L_{ik}(t) = \pi_k \left(1 - e^{-\gamma^*(t,T_i,u_k)}\right), \\ i &= 1, \cdots, n+m; j = 1, \cdots, n; k = 1, \cdots, m. \end{aligned} \tag{37}$$

THEOREM 2. *Under assumptions A1&A2, together with expressions (9) and (10) for α and b, no arbitrage for the bond market implies a unique martingale measure $\mathbb{Q}$ achieved by setting $(g,h) = (\psi, \varphi - 1)$ for the Radon-Nikodým derivative ζ.*

Proof. Let (g,h) be any solution to equation (35) for all $T \geq t$ and $t \geq 0$. For any arbitrary t, it solves particularly the system (35) at $T_1, \cdots, T_{n+m}$

as defined in assumptions A1 & A2 above. This yields

$$\Sigma(t)\,(g(t)-\psi(t))^{\mathsf{T}} + L(t)\left(h(t)-\varphi(t)+\overrightarrow{i}\right)^{\mathsf{T}} = \emptyset, \tag{38}$$

with $h(t) \equiv (h(t,u_k))_{1\times m}, \varphi(t) \equiv (\varphi(t,u_k))_{1\times m}$, and $\overrightarrow{i} \equiv (1,\cdots,1)$. Under the full rank assumption A2, this linear equations model, with $n+m$ equations and $n+m$ unknowns, has a unique solution given by

$$\left(g(t)-\psi(t), h(t)-\varphi(t)+\overrightarrow{i}\right) = \emptyset.$$

This is true for any arbitrary $t \geq 0$. Therefore, $(g,h) = (\psi, \varphi - 1)$,which solves (38), constitutes the unique solution to the infinite dimensional system (35). ∎

Remark 5.1. The full rank assumption reduces to the condition discovered by HJM (1992) for the existence of a unique martingale measure in the presence of pure Brownian uncertainty with $\gamma \equiv 0$.

5.2. The general case with possibly infinite/continuum jump sizes

When the jump size density function has countable infinite or a continuum support, the conditions under which the infinite dimensional system has a unique solution can be also constructed. First of all, we need to assume the following version of full rank condition:

A2'. For all $t \geq 0$, there exists $T_1,\cdots,T_n \geq t$, such that the matrix $\Sigma(t) \equiv \left(\sigma_j^*(t,T_i)\right)_{n\times n}$ is non-singular.

LEMMA 6. *Under assumption A2', equation (35) has a unique solution* $(g,h) = (\psi, \varphi-1)$ *if, and only if, the following linear system*

$$\int_{\mathcal{R}} k(t,T,u)\,\Delta h(t,u)\,\mu(du) = 0, \mathbb{P}\text{-}a.s., \forall T \geq t, \tag{39}$$

has a unique solution $\Delta h = 0, \mu(\cdot)\times\mathbb{P}$*-a.s., where*

$$k(t,T,u) \equiv 1 - e^{-\gamma^*(t,T,u)} - \sigma^*(t,T)\,\Sigma^{-1}(t)\,L(t,u) \tag{40}$$

and $L(t,u) \equiv \left[1-e^{-\gamma^*(t,T_1,u)},\cdots,1-e^{-\gamma^*(t,T_n,u)}\right]^{\mathsf{T}}$.

Proof. For any arbitrary deviation function Δh, we can solve for g from equation (35),

$$g^{\intercal}(t) = \psi^{\intercal}(t) - \Sigma^{-1}(t) \int_{\mathcal{R}} L(t,u)\, \Delta h(t,u)\, \mu(du). \tag{41}$$

Substitute this expression for g back into equation (35) to get (39).

The desired statement follows by the fact that:

$$\Delta h = 0, \mu(\cdot) \times \mathbb{P}\text{-a.s.} \Rightarrow g = \psi, \mathbb{P}\text{-a.s.}.$$

▌

For any given t and $\mathcal{T} \in \mathcal{B}([t,\infty))$, a Borel set, define A: $\mathcal{D}(A) \to \mathcal{L}_2(\mathcal{T}; \lambda(\cdot))$ to be such that,

$$Af : T \mapsto \int_{\mathcal{R}} k(t,T,u)\, f(u)\, \mu(du), \tag{42}$$

where the domain $\mathcal{D}(A)$ of the linear operator A is to contain all $\mu(\cdot)$-square integrable functions such that the right hand side of (42) is to take a finite value for all $T \in \mathcal{T}$; that is,

$$\mathcal{D}(A) \equiv \{f \in \mathcal{L}_2(\mathcal{R}; \mu(\cdot)) \text{ and } Af < \infty, \mathbb{P}\text{-a.s.}, \forall T \in \mathcal{T}\}. \tag{43}$$

LEMMA 7. *For any arbitrary $t \geq 0$ and $\mathcal{T} \in \mathcal{B}([t,\infty))$. If $Af = 0$ has a unique solution $f = 0$ on $\mathcal{T}$, then the linear system (39) admits a unique solution given by $\Delta h = 0, \mu(\cdot) \times \mathbb{P}$-a.s..*

Proof. Any solution to the linear system (39) will necessarily solve the sub-linear system by restricting $T \in \mathcal{T}$. Therefore, if the solution to $Af = 0$ on $\mathcal{T}$ is unique ($f = 0$), then the linear system (39) must have a unique (zero) solution. ▌

The question becomes: Under what conditions will the linear system $Af = 0, \forall T \in \mathcal{T}, \mathbb{P}$-a.s., admit a unique solution $f = 0, \mu(\cdot) \times \mathbb{P}$-almost surely?

A3. Function $k(t,\cdot,\cdot) : [t,\infty) \times \mathcal{R} \times \Omega \to \mathcal{R}$ is assumed to satisfy the following:

(a) For all $t \geq 0$ there exists a $\mathcal{T} \in \mathcal{B}([t,\infty))$, such that $k(t,\cdot,\cdot) \in \mathcal{L}_2(\mathcal{T} \times \mathcal{R}; \lambda(\cdot) \times \mu(\cdot)), \mathbb{P}$-a.s., where $\lambda(\cdot)$ is the Lebesque measure on $\mathcal{T}$.

(b) For all $x \in \mathcal{L}_2(\mathcal{T}; \lambda(\cdot))$,

$$A^{*}x \equiv \int_{\mathcal{T}} k(t,s,\cdot)\, x(s)\, ds \in \mathcal{L}_2(\mathcal{R}; \mu(\cdot)), \mathbb{P}\text{-a.s.}, \tag{44}$$

where A^* is the dual of A.[10]

(c) For the fixed t and $\mathcal{T}$, $A^* : \mathcal{L}_2(\mathcal{T}; \lambda(\cdot)) \to \mathcal{L}_2(\mathcal{R}; \mu(\cdot))$ is surjective; that is, for all $f \in \mathcal{L}_2(\mathcal{R}; \mu(\cdot))$, there exists an $x \in \mathcal{L}_2(\mathcal{T}; \lambda(\cdot))$ such that $f = A^* x$.

Remark 5.2. Conditions A3-(a) and (b) are regularity conditions. Condition A3-(c) is crucial for the uniqueness of a solution. For example, for the special case of finite jump sizes studied in the previous section, conditions A2' and A3-(c) together imply the full rank assumption A2.

PROPOSITION 5. *Under assumptions A2' & A3, the infinite dimensional linear system (35) has a unique solution:* $(g, h) = (\psi, \varphi - 1), \mu(\cdot) \times \mathbb{P}$*-a.s..*

Proof. Let $\Delta h \in \mathcal{D}(A)$ be an arbitrary solution to $Af = 0, \forall T \in \mathcal{T}$. First, we show that, for all arbitrary $f \in \mathcal{L}_2(\mathcal{R}; \mu(\cdot))$,

$$\langle f, \Delta h \rangle_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} \equiv \int_{\mathcal{R}} f(u)\, \Delta h(t, u)\, \mu(du) = 0, \mathbb{P}\text{-a.s.}, \tag{45}$$

where $\langle \cdot, \cdot \rangle$ is the inner product for $\mathcal{L}_2(\mathcal{R}; \mu(\cdot))$: For the given f, by assumption A3, there exists a square-integrable function $x(t, \cdot) \in \mathcal{L}_2(\mathcal{T}; \lambda(\cdot))$ such that $f(\cdot) = A^* x(t, \cdot)$. We have,

$$\langle f, \Delta h \rangle_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} = \langle A^* x, \Delta h \rangle_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} = \langle x, A\Delta h \rangle_{\mathcal{L}_2(\mathcal{T};\lambda(\cdot))} = 0, \mathbb{P}\text{-a.s.},$$

since, by assumption, $A\Delta h(t, s) = 0, \mathbb{P}$-a.s., for all $s \in \mathcal{T}$.

Setting $f = \Delta h \in \mathcal{L}_2(\mathcal{R}; \mu(\cdot))$, it yields

$$\langle \Delta h, \Delta h \rangle_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} = \| \Delta h \|^2_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} = 0, \mathbb{P}\text{-a.s.}.$$

Therefore, we have $\Delta h(t, \cdot) = 0$, $\mu(\cdot) \times \mathbb{P}$-a.s., which, by Lemma 5, constitutes also the unique solution to the linear system (39).

By Lemma 4, we have $g = \psi, \mathbb{P}$-a.s., and the linear system (35) has a unique solution. These end the proof. ▌

As a summary to the above observations, we have:

THEOREM 3. *Under assumptions A2' & A3, together with expressions (9) and (10) for* α *and* b*, no arbitrage for the bond market implies an*

[10] Let E and F be two Hilbert/Banach spaces. Let $A : X \subseteq E \to F$ be a linear operator. The dual of A, denoted as $A^* : F \to E$, is defined to be such that,

$$\langle A^* y, x \rangle_E \equiv \langle y, Ax \rangle_F, \text{ for all } x \in X, y \in F.$$

unique martingale measure $\mathbb{Q}$, *the Radon-Nikodým derivative* ζ *of which is achieved by setting* $(g, h) = (\psi, \varphi - 1)$.

The following establishes a necessary and sufficient condition for condition A3-(c):

PROPOSITION 6. *Consider* $A^* : \mathcal{L}_2(\mathcal{T}; \lambda(\cdot)) \to R(A^*) \subseteq \mathcal{L}_2(\mathcal{R}; \mu(\cdot))$ *as defined above. We have:* $R(A^*) = \mathcal{L}_2(\mathcal{R}; \mu(\cdot))$ *if, and only if, the following conditions are satisfied:*

(i)$R(A^*)$ *is closed in* $\mathcal{L}_2(\mathcal{T}; \lambda(\cdot))$
(ii)$R(A^*)^{\intercal} = \{0\}$; *that is,*

$$\langle A^* x, f \rangle_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} = 0, \forall x \in \mathcal{L}_2(\mathcal{T}; \lambda(\cdot)) \Longrightarrow f = 0, \mu(\cdot) \times \mathbb{P}\text{-}a.s..$$

Proof. See Zeidler (1985), Volume II/A, Theorem 19.K. ▮

Remark 5.3. Condition (i) is purely mathematical, which is satisfied, for example, if there exists a positive constant c, which may depend on $(t, \mathcal{T}; \omega)$, such that

$$\| A^* x \|_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} \geq c \| x \|_{\mathcal{L}_2(\mathcal{T};\lambda(\cdot))}$$

(see Zeidler 1985, Volume II/A, Corollary 19.59).

Remark 5.4. Condition (ii) is necessary for the existence of a unique (zero) solution to $Af = 0, T \in \mathcal{T}$. To see this, suppose to the contrary that condition (ii) is violated, then there exists a $f \in \mathcal{L}_2(\mathcal{R}; \mu(\cdot))$ with $\| f \|_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} > 0$ such that

$$\langle A^* x, f \rangle_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} = 0, \forall x \in \mathcal{L}_2(\mathcal{T}; \lambda(\cdot)), \mathbb{P}\text{-a.s.}.$$

We have,

$$\langle x, Af \rangle_{\mathcal{L}_2(\mathcal{T};\lambda(\cdot))} = \langle A^* x, f \rangle_{\mathcal{L}_2(\mathcal{R};\mu(\cdot))} = 0, \forall x \in \mathcal{L}_2(\mathcal{T}; \lambda(\cdot)), \mathbb{P}\text{-a.s.}.$$

This implies $Af = 0, \mathbb{P}$-a.s.,$\forall T \in \mathcal{T}$. That is, the linear system (39) has a non-zero solution!

6. CONCLUDING REMARKS

The implications of no arbitrage conditions on the dynamics of bond prices and the term structure of interest rates were explored. The paper generalizes HJM (1992) by introducing Lévy jumps into the forward rate process. The coefficients of the forward rate process that are consistent with the no-arbitrage restrictions are fully characterized, as are those of the bond prices. We have also explored the conditions under which a unique risk neutral measure can be constructed from the bond prices.

These problems are independently studied by Bjork, Kabanov and Runggaldier (1997) and Bjork, Di Masi, Kabanov and Ruggaldier (1997). The results reported in Sections 2, 3 and 4 are in line with their findings with minor differences in technical treatments and proofs. Conditions for the existence of unique risk neutral probability measure reported in Section 5 is different to their treatment since we restrict h-function to lie in $\mathcal{L}_2$, the space of square integrable functions (see assumption A3). There are certain advantages associated with this topological treatment. For example, any function in $\mathcal{L}_2$, as a Hilbert space, can be expanded as a linear combination with respect to a sequence of elementary functions, called the 'base' of the Hilbert space. This makes it possible to compute the numerical approximation of the boundle (g, h) by solving a finite dimensional multi-variate linear equations model.

Given a risk neutral probability measure, and given the motions for bonds, forward rates and short term interest rates under such a risk neutral probability measure, we can determine the prices of all income sensitive derivative products. Therefore, the difficulty for pricing derivative products remains when the bond market fails to determine a unique risk neutral measure. This occurs even when coefficients associated with these economic variables are known, and are fully consistent with no arbitrage restrictions. To resolve this difficulty, one may apply the equilibrium approach following CIR (1985a,b) and others.

Concerning the empirical implementation of the HJM's approach, this study shows that, to estimate the risk neutral probability measure it may not be sufficient to estimate coefficients for the term structure of interest rates and forward rates. This is because the linear system (39) under the estimated coefficients may fail to generate a unique zero solution. This is in contrast to economies with pure Brownian uncertainty originally studied by HJM (1992).

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