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Portfolio Selection with Little Information about the Future

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In multiperiod portfolio selection one faces the problem of choosing a proper multiperiod utility function, to determine the planning horizon and the number of future opportunities as well as the financial consequences of these opportunities. In this paper an alternative approach is proposed where the initial portfolio value is required to follow some given growth pattern. Following this approach it is shown that under reasonable assumptions a solution can be found that requires neither a utility function nor information about the planning horizon, the required growth rates, the number of future opportunities or the financial consequences of these opportunities.

Key Words: Multiperiod; Portfolio; Selection.
JEL Classification Number: G11

1. INTRODUCTION

To solve portfolio problems it is required to estimate the financial consequences of portfolio decisions and to formulate appropriate decision criteria. Both requirements cause substantial diffculties for multiperiod portfolio problems. In particular, if one follows the standard expected utility maximization approach a proper multiperiod utility function and the probability distribution of multiperiod cashflows have to be determined. This hardly can be done.

Clearly under such circumstances it would be helpful to have decision criteria that are not based on expected utility maximization and that can be applied even if the distribution of multiperiod cashflows is (completely or in part) unknown.

Hellwig (2004), Hellwig, Speckbacher and Wentges (2000), Korn (1997, 1998, 2000), Korn and Schäl(1999) and Speckbacher(1998) analyse an approach to portfolio selection that is not based on expected utility maximization. According to this approach a portfolio is required to satisfy two conditions. First, the portfolio should be efficient. That is, it should

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maximize the present value of future cashflows. Second, the present value should follow a given growth pattern according to the growth preferences of the investors. A portfolio that satisfies these conditions is called growth-oriented.

Hellwig et al (2000) and Hellwig (2002) show that the approach is incompatible with utility maximization in the following sense. Suppose that a growth-oriented portfolio has been found. The efficiency requirement implies the existence of an increasing and concave utility function such that the growth-oriented portfolio is maximizing utility. Now assume that the set of feasible portfolios is changed. Then applying the same utility function generally leads to a portfolio that is not growth-oriented. This contradicts the utility maximizing approach where the utility function is assumed to be independent from the feasible set.

In the following it is shown that under reasonable assumptions a growth-oriented portfolio can be found even if there is little information about the future.

2. THE DECISION PROBLEM

The following analysis is based on a standard event tree approach (e.g. Huang/Litzenberger,1988).

$S = \{0, \ldots, n\}$ denotes the set of events (nodes), S_t the set of events at time t where $t = 0, \ldots, N$ and $S_0 = \{0\}, N(s)$ the successors of s, $F(s)$ the immediate successors of s and s^- the uniquely given immediate predecessor of s.

There are m investment and/or financing opportunities. $x = (x_1, \ldots, x_m)^T$ denotes the activity level of these opportunities, $A \in \mathbb{R}^{(n+1)\times m}$ the payoff-matrix, $b = (b_0, 0, \ldots, 0)^T \in \mathbb{R}^{n+1}$ the vector of endowments ($b_0 > 0$) and $c = (0, c_1, \ldots, c_n)^T$ a consumption vector where it is assumed that b_0 is totally invested.

Then a growth oriented portfolio $\overline{x}$ is defined as a portfolio $\overline{x}$ that satisfies the following conditions for some vector of positive prices $\overline{p} = (1, \overline{p}_1, \ldots, \overline{p}_n)$ (Hellwig, 2004).

(C1) $\overline{p}$ supports $(\overline{x}, \overline{c})$, that is $(\overline{x}, \overline{c})$ is an optimal solution of

$$\overline{V}_0 = V_0(\overline{c}, \overline{p}) := \max\{\overline{p}c | c \in C\}$$

where $C = \{c | c = Ax + b, x \geq 0\}$.

(C2) $\overline{p}$ is compatible with the required growth of the initial value $\overline{V}_0$ that is

$$\overline{V}_s = V_s(\overline{c}, \overline{p}) = (1 + \alpha_s)\overline{V}_{s^-}, s = 1, \ldots, n$$

where $V_s(c,p) = \frac{1}{p_s}\sum_{k\in N(s)} p_k c_k = \frac{1}{p_s}\sum_{k\in F(s)} p_k(V_k + c_k)$ is the portfolio value in s and α_s is the required growth rate of the portfolio value between s^- and s.

Neither (C1) nor (C2) require information about the probability distribution of the cashflows. Here the following information is assumed to be available to the investor.

(A1) The investor estimates the probability for event $s \in S_1$ to be $\pi_s > 0$ where $\sum_{s\in S_1} \pi_s = 1$.

This assumption seems reasonable because probability beliefs about events in the near future generally are more reliable than probability beliefs about events in the more distant future.

Given (A1) one can decompose $\overline{p}_s (s \in S_1)$ into π_s and a risk-adjusted discount factor $\overline{q}_s : \overline{p}_s = \pi_s \cdot \overline{q}_s = \pi_s/(1+\overline{r}_s)$. $\overline{r}_s$ can be understood as the required return if event s is realized and (C2) can be strengthenedto(Hellwig2004)

(C3) $\frac{\overline{V}_s+\overline{c}_s}{1+\overline{r}_s} = \overline{V}_0$ for all $s \in S_1$. is required.

The formulation of the feasible set C allows to consider different opportunities in every node, short-selling (by appropriately defining the opportunities) and multi-period as well as single-period opportunities. Here the following assumption is made.

(A2) At time zero only single-period opportunities are considered.

Assumption (A2) underlines most portfolio selection models such as the Markowitz model.

It will be shown in the following that (A1) and (A2) imply the existence of a uniquely given portfolio decision at time zero such that (C1) - (C3) are satisfied.

This decision does not depend

- upon the growth rates $\alpha_s, s = 1, \ldots, n$,
- upon the time horizon N,
- upon the number of events after time $t = 1$,
- or upon the opportunities that can be realized from time $t = 1$ onwards, their financial consequences and the probability distribution of these consequences.

3. THE SOLITION

Before presenting the main result some preliminary remarks are helpful. Let $(\overline{p}, \overline{x}, \overline{c})$ satisfy (C1) - (C3) where assumptions (A1) and (A2) are met.

Then $K_i = \sum_s a_{is}\overline{p}_s$ can be understood as the present value of opportunity i. (C1) implies $K_i \leq 0$ for all i and $\overline{x}_i = 0$ if $K_i < 0$. As a consequence $\overline{V}_0 = b_0$.

(A2) implies $\overline{V}_s = -\sum_{i \in I_s} a_{is}\overline{x}_i (s \in S_1)$ where I_s denotes the set of opportunities that can be realized in s. It follows that the cashflow $f_s(\overline{x}) = \sum_{i \in I_0} a_{is}\overline{x}_i$ that results from the portfolio decision at time zero in $s \in S_1$ is given by $f_s(\overline{x} = \overline{c}_s + \overline{V}_s$. Thus (C3) can be written as $\frac{\overline{V}s + \overline{c}_s}{1+\overline{r}_s} = \frac{f_s(\overline{x})}{1+\overline{r}_s} = \overline{V}_0 = b_0, s \in S_1$.

THEOREM 1. *Let $(\overline{p}, \overline{x}, \overline{c})$ satisfy (C1) - (C3) and let assumptions (A1) and (A2) be met. Then $\overline{x}_i, i \in I_0$, is an optimal solution of (P):*
$\max\{\sum_{s \in S_1} \pi_s \ln f_s | \sum_{i \in I_0} a_{io}x_i = b_0, f_s = \sum_{i \in I_0} a_{is}x_i$ *for all* $s \in S_1, x_i \geq 0, i \in I_0\}$.

Proof.
The necessary and sufficient conditions for a portfolio x^* to be optimal for (P) is that there exists $w = (w_0, \ldots, w_n)$ such that

$$\sum_{s \in S_1} a_{is}w_s + a_{io}w_0 \ = \ : W_i \leq 0, x_i^* \geq 0, (i = 1, \ldots, m)$$

$$W_i < 0 \Rightarrow x_i^* = 0, i \in I_0 \quad (1)$$

$$\pi_s / f_s(x^*) \ = \ w_s, s \in S_1 \quad (2)$$

Define $w_s := \overline{p}_s / b_0 = \frac{\pi_s}{(1+\overline{r}_s)b_0}$. Then $\frac{f_s(\overline{x})}{1+\overline{r}_s} = f_s(\overline{x})w_s b_0 / \pi_s = b_0$ or $\pi_s / f_s(\overline{x}) = w_s$. Thus (2) is satisfied. Because $W_i = \frac{1}{b_0}K_i$ the same holds for (1). Therefore $\overline{x} = x^*$ is the uniquely given optimal solutionof (P). ▮

As an example assume three events at $t = 1$ and two investment opportunities P_1 and P_2 with the unit cashflows $P_1 : (-1; 1.1; 0.95; 1.05), P_2 : (-1; 0.9; 1; 1.15)$. For $b_0 = 100, \pi_1 = 0.3, \pi_2 = 0.2, \pi_3 = 0.5$ the optimal solution of (P) is $\overline{x}_1 \approx 76.54, \overline{x}_2 \approx 46.46$, $f_1(\overline{x}) \approx 105.31, f_2(\overline{x}) \approx 96.17, f_3(overlinex) \approx 107.35$. The associated rates of return are $\overline{r}_1 \approx 5.31\%, \overline{r}_2 \approx -3.83\%, \overline{r}_3 \approx 7.35\%$. Assume now that event two is realized at the end of the first period. Then, based on the information that the investor has at that point of time, he has to decide upon the growth rate α_2. α_2 is realized if $\overline{c}_2 = 100(\overline{r}_2 - \alpha_2) \approx 100(-0.0383 - \alpha_2)$. Thus if $\alpha_2 = \overline{r}_2 = -3.83\%$ the investor can consume nothing. If $\alpha_2 < \overline{r}_2$ he can consume a positive amount and if $\alpha_2 > \overline{r}_2$ he has to make an additional payment.

4. FINAL COMMENTS

1. The preceding analysis is based on the definition of value as present value after consumption. The conclusions remain the same if the value is defined as present value including consumption i.e. $V_s(c,p) = \frac{1}{p_s}\sum_{k \in N(s)} p_k c_k + c_s$ for all s.

2. The conclusions also remain the same if there are upper bounds for the activities $i \in I_0$. Such bounds may lead to an increase of the initial value. In order to see this assume for the above example that at most 10 monetary units can be borrowed at an interest rate of 4%. Solving (P) with this additional opportunity results in an optimal program $\overline{x}_1 \approx 84.12, \overline{x}_2 \approx 25.88, \overline{x}_3 = 10$ where $\overline{x}_3$ is the amount borrowed. The associated rates of return are $\overline{r}_1 \approx 5.39\%, \overline{r}_2 \approx -4.63\%, \overline{r}_3 \approx 7.66\%$. The present value of the amount borrowed is $K_F \approx 0.0029$ and the initial value is $\overline{V}_0 = 100 + K_F \approx 100.0029$.

REFERENCES

Hellwig, K., G. Speckbacher, and P. Wentges, 2000. Utility maximization under capital growth constraints. *Journal of Mathematical Economics* **33**,1-22.

Hellwig, K., 2002b. Growth and utility maximization. *Economics Letters* **77**, 377-380.

Hellwig, K., 2004. Portfolio selection subject to growth objectives. *Journal of Economic Dynamics and Control* **28**, 2119-2128.

Huang, C. F. and R. H. Litzenberger, 1988. *Foundations for Financial Economics.* North-Holland, New York-Amsterdam-London.

Korn, R., 1997. Value preserving portfolio stratetgies in continuous-time models. *Mathematical Methods of Operations Research* **45**, 1-42.

Korn, R., 1998. Value preserving portfolio strategies and the minimal martingale measure. *Mathematical Methods of Operations Research* **47**, 169-179.

Korn, R. and M. Schäl, 1999. On value preserving and growth optimal portfolios. *Mathematical Methods of Operations Research* **50**, 189-218.

Korn R., 2000. Value preserving strategies and a general framework for local approaches to optimal portfolios. *Mathematical Finance* **10**, 227-241.

Speckbacher, G., 1998, Maintaining capital intact and WARP. *Mathematical Social Sciences* **36**, 145-155.