ANNALS OF ECONOMICS AND FINANCE **8-2**, 397–414 (2007)

Asset Prices and Hyperbolic Discounting

Liutang Gong

Guanghua School of Management, Peking University, Beijing, 100871, China
Institute for Advanced Study, Wuhan University, Wuhan, 430072, China

William Smith

Department of Economics, Fogelman College of Business & Economics
University of Memphis
E-mail: wtsmith@memphis.edu

and

Heng-fu Zou

Guanghua School of Management, Peking University
Institute for Advanced Study Wuhan University
The World Bank, Washington DC, 20433,USA

This paper explores the implications of hyperbolic discounting for asset prices and rates of return. Hyperbolic discounting has no effect on the equity premium. However, by making people less patient, causes stock prices to be lower, and interest rates higher, than with exponential discounting. In addition, hyperbolic discounting dampens the marginal effect of risk on stock prices, relative to the exponential case.

Key Words: Asset-Pricing; Hyperbolic Discounting.
JEL Classification Numbers: D91, E21, G11.

1. INTRODUCTION

Since Laibson's (1994, 1997a,b) path-breaking work, economists have probed deeply into the implications of dynamically inconsistent preferences [see Ainslie and Haslam (1992), Barro (1999), Bernheim, Ray, and Yeltekin (2000), Gul and Pesendorfer (2001), Harris and Laibson (2001a, b; 2003),

1529-7373/2006

Krusell and Smith (2001) to mention just a few important papers][1] Recently, one branch of the literature has focused on how dynamically inconsistent preferences affect consumption and portfolio behavior under conditions of uncertainty [Harris and Laibson (2001b), Luttmer and Mariotti (2003)]. In particular, Palacios-Huerta (2003) has adapted Merton's (1969, 1971) classic model of consumption and portfolio choice to incorporate hyperbolic discounting. This is a particularly appealing framework because it permits a clear picture of how hyperbolic discounting alters consumer behavior under uncertainty.

In Gong, Smith, and Zou (2006) we have employed Palacios-Huerta's model to explore the comparative statics of risk under hyperbolic discounting. With exponential discounting and constant-relative-risk-aversion (CRRA) utility consumption is a linear function of risk. With hyperbolic discounting, however, the rate of time preference becomes endogenous, so that risk affects consumption in a non-linear way. In particular, hyperbolic discounting amplifies the marginal effect of risk on consumption, relative to the exponential case. This means that it is not true — as often asserted — that hyperbolic discounting and exponential discounting are observationally equivalent. It is true that the level of consumption predicted by a hyperbolic model can be matched by imputing a higher rate if time preference to an exponential model. However, the two models offer radically different comparative static predictions.

In this paper we expand the model to investigate the implications of hyperbolic discounting for asset prices and rates of return. We incorporate the Palacios-Huerta model of consumption with hyperbolic discounting into a equilibrium asset-pricing model á la Lucas (1978). Hyperbolic discounting makes people less patient. This depresses savings and reduces the demand for stocks, so that stock prices fall and interest rates increase. Furthermore, hyperbolic discounting dampens the marginal effect of risk on stock prices, relative to the exponential case.

Two other papers have studied consumption behavior in continuous-time with risky assets and hyperbolic discounting. Harris and Laibson (2001b) work in continuous-time in order to avoid the "pathologies" that crop up indiscrete-time models of with hyperbolic discounting.[2] They establish general existence results and prove that consumption is continuous and monotonic in wealth. Luttmer and Mariotti (2003) consider the continuous-time approximation of a discrete-time consumption/portfolio model with hyperbolic discounting. Like Palacios-Huerta (2002), they show that hyperbolic

[1]Gul and Pesendorfer (2002) develop a model with dynamically consistent preferences.

[2]In discrete time, consumption may be discontinuous and non-monotonic in wealth and there may be multiple equilibria. See Laibson (1997b), Morris and Postlewaite (1997), O'Donoghue and Rabin (1999), Harris and Laibson (2001b), and Krusell and Smith (2000).

discounting affects consumption and the risk-free rate, but does not alter portfolio demands or excess returns. They do not investigate the comparative statics of consumption or of asset prices.

2. CONSUMPTION AND PORTFOLIO POLICIES

Following Palacios-Huerta (2002), imagine a consumer who has an infinite planning horizon and maximizes expected lifetime utility. He exhibits quasi-hyperbolic discounting, so that his discount function is

$$\begin{array}{ll} e^{-\theta s}, & t \leq s \leq t+h, \\ \delta e^{-\theta s}, & t+h \leq s < \infty. \end{array} \tag{1}$$

Beginning at time t the discount function decays exponentially at the constant rate θ until time $t+h$. At time $t+h$ it drops discontinuously by a fraction $\delta \in (0,1]$; thereafter it continues to decay at the rate δ. This subsumes two important, special cases: If $\delta = 1$ we recover Merton's (1969, 1971) exponential discounting, while if $h \to 0$ there is "instantaneous gratification," proposed by Harris and Laibson (2001b).

The consumer has time-separable utility with constant relative risk aversion (CRRA). Expected lifetime utility is thus

$$E_t U_t = E_t \int_t^{t+h} e^{-\theta s} \frac{C_s^{1-\gamma}}{1-\gamma} ds + \delta \int_{t+h}^{\infty} e^{-\theta s} \frac{C_s^{1-\gamma}}{1-\gamma} ds, \tag{2}$$

where $\gamma > 0$ is the coefficient of relative risk aversion. Intuitively, Equation (2) says that the "current self" makes decisions from time t to time $t+h$, whereupon the "next self" starts to make the decisions.

There are two assets. A riskless asset pays a constant rate of return r. A risky asset has a price P_t that follows a geometric Brownian motion,

$$\frac{dP_t}{P_t} = \mu dt + \sigma dZ_t, \tag{3}$$

where Z_t is a Wiener process. Define λ_t as the share of wealth W_t invested in the risky asset. The budget constraint is

$$dW_t = \{[(1-\lambda_t)r + \lambda_t \mu] W_t - c_t\} dt + \sigma \lambda_t W_t dZ_t. \tag{4}$$

The consumer's problem is to choose policies λ_t and C_t to maximize Equation (2) subject to Equation (4), given initial wealth W_0. Palacios-Huerta (2003) shows that the optimal policies for this problem are

$$\lambda_t^* = \frac{\mu - r}{\gamma \sigma^2}, \tag{5}$$

$$C_t^* = c_H W_t, \tag{6}$$

where the marginal propensity to consume (MPC) c_H is determined implicitly by the equation

$$c_H = \frac{\lfloor \theta + (1-\delta)c_H e^{-\theta h} E_0(W_h/W_0)^{1-\gamma} \rfloor - (1-\gamma)\lfloor \mu_w - \gamma\sigma_w^2/2 \rfloor}{\gamma}, \quad (7)$$

and $\mu_w = (1-\lambda^*)r + \lambda^*\mu$ and $\sigma_w^2 = {\lambda^*}^2\sigma^2$ are the optimal mean and variance of the rate of return to the portfolio. The subscript "H" denotes "hyperbolic".

The portfolio demand in Equation (5) is exactly the same as in Merton (1969, 1971). Therefore, as Palacios-Huerta (2002) emphasizes, hyperbolic discounting has no effect on portfolio demands.

To understand the consumption function in Equations (6) and (7) it is useful to consider the MPC for the exponential benchmark in Merton (1969, 1971):

$$c_M = \frac{\theta - (1-\gamma)\lfloor \mu_w - \gamma\sigma_w^2/2 \rfloor}{\gamma}. \quad (8)$$

The subscript "M" stands for "Merton". The term in braces in Equation (8) is the certainty-equivalent rate of return to the portfolio. An increase in risk lowers the certainty-equivalent rate of return, which then increases or decreases consumption depending upon whether relative risk aversion γ, is less than or greater than one.[3] The essential thing to note is that, in the presence of exponential discounting, consumption is a linear function of the constant rate of time preference and the certainty-equivalent rate of return.

Now compare Equations (7) and (8). It is clear on inspection that that hyperbolic discounting ($\delta < 1$) has the effect of increasing the rate of time preference from θ to $\theta + (1-\delta)c_H e^{-\theta b} E_0(W_h/W_0)^{1-\gamma}$. Intuitively, the "current self" anticipates that the "next self" will consume too much and so attaches less value at the margin to future consumption. [Harris and Laibson (2001a)].[4] Hyperbolic discounting raises consumption, relative to the exponential benchmark, by making people less patient.

[3]Since Weil (1989) we have known that it is really intertemporal substitution, rather than risk aversion, that governs the sign of the effect of risk on consumption. It would be straightforward to develop a version of this model with Generalized Isoelastic (GIE) preferences [Epstein (1987), Epstein and Zin (1989, 1991), Duffie and Epstein (1993a, b), Svensson (1989), Weil (1989)] in order to disentangle risk aversion from intertemporal substitution. However, doing so would not add much to the point here, and would distract attention from the time-separable benchmark used by Palacios-Huerta (2002). The reader should feel free to interpret the coefficient attached to the certainty-equivalent rate of return in Equations (8) and (9) as $1 - 1/\varepsilon$, where ε is the intertemporal elasticity of substitution for riskless consumption paths.

[4]Our effective rate of time preference corresponds to the effective discount factor in Harris and Laibson (2001a).

At first glance this would seem to suggest that hyperbolic discounting is observationally equivalent to exponential discounting: it is always possible to match the level of consumption predicted in a hyperbolic model by calibrating an exponential model to have a higher discount rate. Indeed, Palacios-Huerta (2003) asserts for this reason that the canonical model of consumption and portfolio choice remains “intact” after the introduction of hyperbolic discounting. This mirrors Barro’s (1999) argument that the neoclassical growth model remains “intact” after introducing a non-constant rate of time preference.

Gong, Smith, and Zou (2006) rebut this argument. Even if the two models can be calibrated to generate the same level of consumption, they still may make very different comparative static predictions. Notice that the rate of time preference with hyperbolic discounting [in Equation (7)] depends upon the expected growth in wealth between period 0 and period h. This is a manifestation of the general result of Harris and Laibson (2001b): the value function for a consumer with dynamically inconsistent, hyperbolic preferences is the same as to the value function for a consumer with dynamically consistent, exponential preferences and a wealth-dependent utility function. In other words, hyperbolic discounting induces an “indirect” form (i.e., through the value function) of the “spirit of capitalism” — the old notion [Weber (1958)] that people may derive utility from wealth itself, in addition to consumption — that has recently been used to explain asset prices [Bakshi and Chen (1996), Smith (2001),Gong and Zou (2002a)].[5] In this literature, the level of wealth (or some other measure of status) yields utility. With hyperbolic discounting it is the growth of wealth that matters, rather than the level. This is similar in spirit to an idea originally espoused by Marshall (1979), and recently explored by Gootzeit, Schneider, and Smith (2002), that people derive utility from the act of saving, from the accumulation of wealth rather than the level of wealth.

Using the fact that wealth is log-normal, it is straightforward to calculate

$$E_0(W_h/W_0)^{1-\gamma} = e^{(1-\gamma)[\mu_w - c_H - \gamma\sigma_w^2/2]h}. \tag{9}$$

Changes in the mean and the variance of the rate of return, as well as changes in the MPC itself, alter the effective rate of time preference exponentially. In other words, the rate of time preference is endogenous. This will have profound implications for the comparative statics of the model.

Following Gong, Smith, and Zou (2006), consider how does uncertainty affects consumption in the presence of hyperbolic discounting. To simplify exposition, and to set the stage for the next section, we will assume that

[5]The spirit of capitalism has also been used to explain saving [Zou (1995)] and growth [Zou (1994), Smith (1999), Gong and Zou (2002a,b)].

$\lambda^* = 1$.[6] In this case, the marginal propensity to consume is determined implicitly by

$$c_H = \frac{\lfloor \theta + (1-\delta)c_H e^{-\theta h + (1-\gamma)[\mu - c_H - \gamma\sigma^2/2]h} \rfloor - (1-\gamma)\lfloor \mu - \gamma\sigma^2/2 \rfloor}{\gamma}. \tag{10}$$

Given the transversality condition, there is a unique value of c_H that solves this equation.[7] Note that the rate of time preference in Equation (10) is an increasing function of the MPC. This is similar to Harris and Laibson (2001a), where the current discount factor is a decreasing function of the future MPC. The MPC in Equation (10) is the fixed-point that captures the dependence of the current MPC on the future MPC.

Now consider how risk affects consumption. In the benchmark case with exponential discounting we have seen that consumption will increase or decrease linearly with σ^2 depending upon whether relative risk aversion is less than or greater than one,

$$\partial c_M / \partial \sigma^2 = (1-\gamma)/2. \tag{11}$$

In the general case with hyperbolic discounting we find

$$\frac{\partial c_H}{\partial \sigma^2} = (1-\gamma)\frac{\gamma}{2}\frac{1-(1-\delta)hc_H e^{-\theta h}E_0(W_h/W_0)^{1-\gamma}}{\gamma + [(1-\gamma)c_H h - 1](1-\delta)e^{-\theta h}E_0(W_h/W_0)^{1-\gamma}}. \tag{12}$$

The direction of the effect of risk on consumption still depends upon the magnitude of relative risk aversion. However, with hyperbolic discounting risk no longer has a simple linear effect of risk on consumption.

In Gong, Smith, and Zou (2006) we demonstrate

PROPOSITION 1. *The absolute value of the marginal effect of risk on consumption is greater under hyperbolic discounting than under exponential discounting:* $|\partial c_H/\partial\sigma^2| > |\partial c_M/\partial\sigma^2|$. *Furthermore, if* h *is sufficiently small and* $b > 1-\delta$, *consumption is a concave function of risk when discounting is hyperbolic:* $\partial^2 c_H/\partial {\sigma^2}^2 < 0$.[8]

Intuitively, consumption still increases or decreases with risk depending upon the magnitude of relative risk aversion. However, hyperbolic discounting amplifies the effect of risk on consumption, relative to the exponential

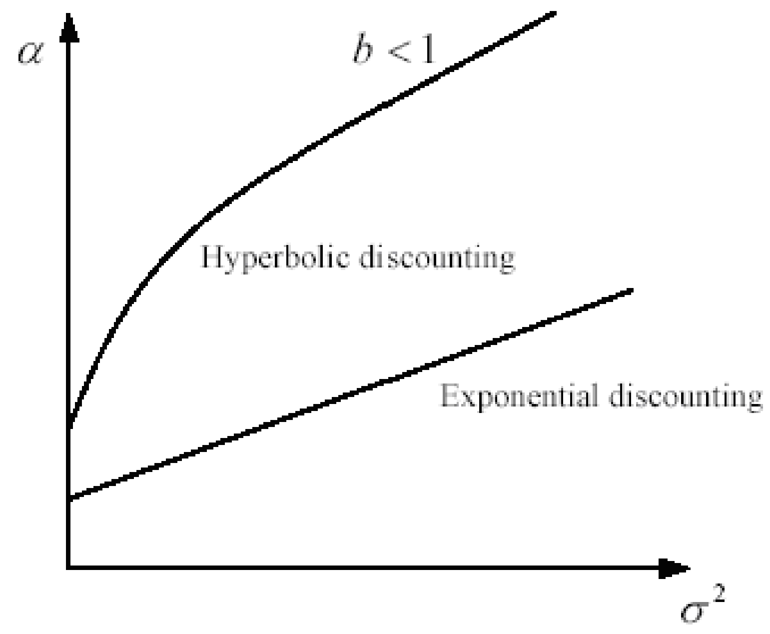
---

[6] In an equilibrium where the riskless asset is in zero net supply $\lambda_t = 1$.

[7] Write the right-hand side of Equation (8) as $RHS(c_H)$. The TVC implies $RHS(0) > 0$. Furthermore, since $\gamma + [(1-\gamma)c_H h - 1](1-\delta)E_0(W_h/W_0)^{1-\gamma} > 0$, it can be shown that $0 < dRHS/dc_H < 1$. Therefore, RHScrosses the 45° line once.

[8] Although is concave for small, nonzero it is not when there is instantaneous gratification, It is plausible to think that consumers usually are able to commit to decisions over short periods.

FIG. 1.

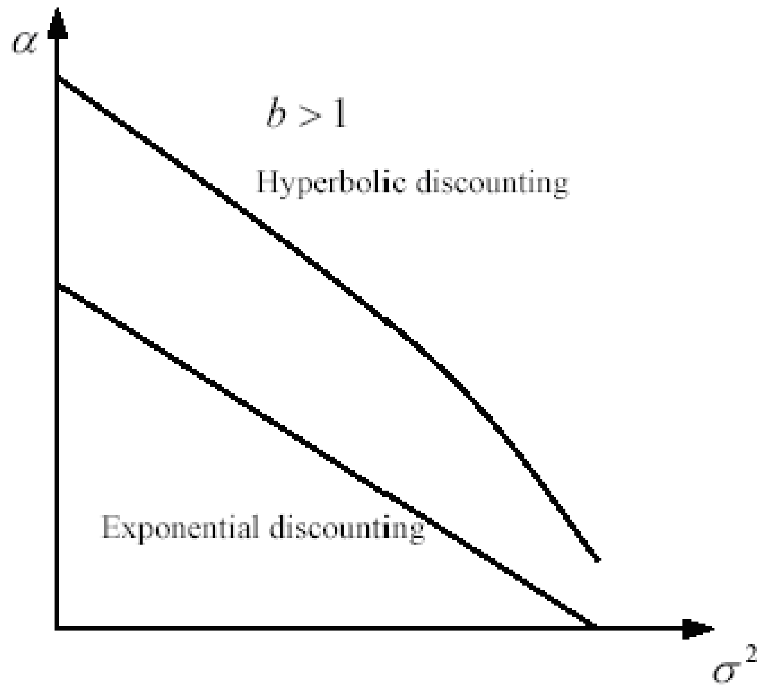


benchmark. Moreover, the effect is no longer linear: consumption increases (or decreases, depending upon γ) at a decreasing rate, as risk increases.

In Figures I and II we show consumption as a function of risk for the cases where γ is less than or greater than unity, respectively. The linear functions depict the exponential benchmarks.

Consider the case when $\gamma < 1$, in Figure I. If discounting is hyperbolic, then consumption is an increasing, concave function of risk, while if discounting is exponential consumption is an increasing linear function of risk.. The hyperbolic consumption function is always steeper than the exponential consumption function. This means that the marginal effect of risk on consumption is greater under hyperbolic discounting than under exponential discounting. However, because the hyperbolic consumption function is concave, the marginal impact of risk on consumption decreases as risk increases. Why, intuitively, is this happening? Under exponential discounting the rate of time preference is just θ; since the rate if time preference is exogenous the MPC increases linearly with risk. Under hyperbolic discounting, however, the "effective" rate of time preferences is $\theta + (1-\delta)c_H e^{-\theta b} E_0(W_h/W_0)^{1-\gamma}$. Because the rate of time preference is

FIG. 2.



endogenous, the increase in the MPC feeds back to raise the effective rate of time preference. This magnifies the increase in consumption caused by the increase in risk. However, the increase in risk also has a direct effect on the effective rate of time preference: for a given c_H an increase in risk lowers the rate of time preference when $\gamma < 1$ [see Equation (9)]. This exerts a countervailing effect on the MPC, the magnitude of which increases as risk increases.

Conversely, consider the case where $\gamma > 1$, in Figure II. Now consumption decreases with in risk: the relationship is again linear in with exponential discounting and concave with hyperbolic discounting. The slope of the function is always more negative in the hyperbolic than in the exponential case. Hence, hyperbolic discounting amplifies the decline in consumption associated with an increase in risk.

3. IMPLICATIONS FOR ASSET PRICES

To develop the implications of hyperbolic discounting for asset pricing, consider the following Lucas (1978) "tree" model. A tree yields "fruit" D_t

(dividends) according to the geometric Brownian motion:

$$\frac{dD_t}{D_t} = \nu dt + \sigma_D dZ_t. \tag{13}$$

Investors can buy shares in the tree (a stock) at the price (ex-dividend) P_t. The supply of shares is inelastic and normalized in size to one. Using the notation in Equation (3), the cum dividend rate of return is then

$$\frac{dP_t}{P_t} + \frac{D_t}{P_t} = \mu dt + \sigma_D dZ_t, \tag{14}$$

where $\mu = \pi + D_t/P_t$ and π is expected capital gains. In equilibrium D_t/P_t will be constant, so that the expected rate of return will also be a constant.

An equilibrium consists of a pricing function $P_t = f(D_t)$ and a risk-free interest rate r such that for $t \in [0, \infty)$

(1) the representative consumer obeys the optimal policies in Equations (5), (6), and (7),

(2) all dividends are consumed, so that $C_t = D_t$ and

(3) the riskless asset is in zero net supply, $\lambda_t = 1$.

In Appendix B we show

PROPOSITION 2. *The equilibrium price function and interest rate are*

$$P_t = A_H D_t, \tag{15}$$
$$r = \nu + 1/A_H - \gamma\sigma_D^2, \tag{16}$$

where

$$A_H = \frac{1 - (1-\delta)e^{-\theta h + (1-\gamma)(\nu - \gamma\frac{\sigma_D^2}{2})}}{\theta - (1-\gamma)(\nu - \gamma\frac{\sigma_D^2}{2})} \tag{17}$$

The subscript "H" again denotes "hyperbolic". Notice that the stock price is proportional to dividends, so that capital gains is equal to the growth rate of dividends; that is, $dP_t/P_t = dD_t/D_t$, so that $\pi = \nu$ and $\sigma^2 = \sigma_D^2$.[9]

Sketch of Proof:

[9]We assume that the denominator in Equation (17) is positive. This is a necessary and sufficient condition for the TVC to be satisfied in the exponential version of the model, discussed below.

In the appendix we demonstrate that the pricing function must satisfy the following, non-linear, second-order differential equation:

$$1-(1-\delta)e^{-thetah+(1-\gamma)([\frac{f'(D_t)D_t}{f(D_t)}+\frac{f''(D_t)D_t}{f(D_t)}]\nu-\gamma[\frac{f'(D_t)D_t}{f(D_t)}]^2\frac{\sigma_D^2}{2})h}$$
$$= f\left[\theta-(1-\gamma)\left(\left[\frac{f'(D_t)D_t}{f(D_t)}+\frac{f''(D_t)D_t}{f(D_t)}\right]\nu-\gamma\left[\frac{f'(D_t)D_t}{f(D_t)}\right]^2\frac{\sigma_D^2}{2}\right)\right] \tag{18}$$

The solution to this equation is given by equations (15) and (17). The equilibrium interest rate in Equation (16) then follows from the proportionality of the asset price to dividends and the fact that $\lambda_t = 1$ in equilibrium.

How does hyperbolic discounting affect asset prices and rates of return? Notice first that in equilibrium all wealth is invested in the stock, so that $\lambda_t = 1$ Using the portfolio demand in Equation (5) and the fact that $\sigma^2 = \sigma_D^2$ it follows that

$$\mu - r = \gamma\sigma_D^2, \tag{19}$$

where $\mu = \nu + 1/A_H$. This implies

PROPOSITION 3. *Hyperbolic discounting has no effect on the equity premium.*

Hyperbolic discounting is of no use to explaining the equity premium paradox, for the simple reason that it does influence portfolio demands.

However, hyperbolic discounting does affect the levels of stock prices and interest rates. To see this, it is useful to consider the exponential benchmark as a special case. When $h = 0$ the equilibrium stock price in equations (15) and (17) reduces to $P_t = A_M D_t$, where

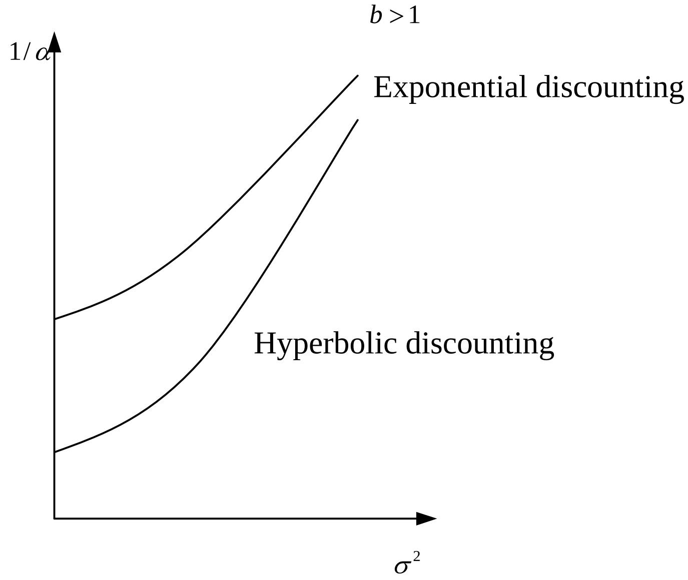
$$A_M = \frac{1}{\theta-(1-\gamma)(\nu-\gamma\frac{\sigma_D^2}{2})} \tag{20}$$

This is the equilibrium stock price that would emerge if the consumption/portfolio model in Merton (1969, 1971) were embedded in a Lucas (1978) equilibrium model, so "M" is again a mnemonic for "Merton". Comparing this to the consumption in Equation (8), it is evident that the stock price is inversely proportional to the MPC.

Now compare Equation (17) and (19). As suggested by Palacios-Huerta (2003), hyperbolic discounting will lower the level of the stock price by raising the discount rate. From Equation (16) this also increases the interest rate, by increasing the dividend/price ratio. Thus.

PROPOSITION 4. *Hyperbolic discounting lowers stock prices and raises the risk-free rate.*

FIG. 3.



Now consider how risk affects the stock price. In the exponential model [Equation (19)] it is immediate that

$$\frac{\partial A_M}{\partial \sigma_D^2} = -(1-\gamma)\frac{\gamma}{2}\frac{1}{\left[\theta-(1-\gamma)\left(\nu-\gamma\frac{\sigma_D^2}{2}\right)\right]^2}, \tag{21}$$

$$\frac{\partial^2 A_M}{\partial \sigma_D^2} = (1-\gamma)^2\frac{\gamma^2}{2}\frac{1}{\left[\theta-(1-\gamma)\left(\nu-\gamma\frac{\sigma_D^2}{2}\right)\right]^3} > 0. \tag{22}$$

In the canonical model an increase in uncertainty about dividend growth will lower the stock price if $\gamma < 1$, and raise it if $\gamma > 1$. Furthermore, from Equation (17), the stock price will be convex in risk. The intuition is straightforward. Suppose that $\gamma > 1$. An increase in risk will then lower the MPC. Since people save more, the demand for the stock increases and its price rises. The stock price increases at an increasing rate because it is inversely proportional to the MPC.

What happens when there is exponential discounting? In Appendix C we prove

PROPOSITION 5. *The absolute value of the marginal effect of risk on the price of the risky asset is greater under hyperbolic discounting than under* *exponential discounting:* $|\partial A_H/\partial\sigma_D^2| < |\partial A_M/\partial\sigma_D^2|$. *Furthermore, if* h *is sufficiently small then the asset price is a convex function of risk when discounting is hyperbolic:* $\partial^2 A_H/\partial{\sigma_D^2}^2 > 0$.

In other words, hyperbolic discounting dampens the marginal effect of risk on stock prices, relative to the effect predicted by the exponential model. Why does this happen? Consider again the empirically plausible case where $\gamma > 1$. This is depicted in Figure III. In the exponential model an increase in risk lowers consumption [Equation (11)]. Since people are saving more, the demand for the risky asset increases, and with it the price of the risky asset [Equation (20)]. This effect also occurs in the hyperbolic model. However, in the hyperbolic model the increase in risk also raises the rate of time preference by changing the expected growth of wealth. Since people are less patient, savings falls by more than in the exponential model, causing the price of the asset to decrease relative to the increase in the exponential model.

4. CONCLUSION

By endogenizing the rate of time preference, hyperbolic discounting introduces a non-linearity into the consumption/portfolio decision. We have shown [Gong, Smith, and Zou (2006)] that this causes the comparative static predictions of the hyperbolic model to differ radically from the exponential model. Hyperbolic discounting amplifies the effect of changes in risk on consumption.

In this paper we have explored the implications of this non-linearity for asset prices and rates of return. Hyperbolic discounting does not affect the equity premium. However, it does alter the way in which the level of stock prices and interest rates are affected by risk. Hyperbolic discounting induces people to save less than in the exponential case, lowering the demand for stocks. This lowers stock prices and raises the risk-free rate. In addition, hyperbolic discounting reduces the marginal effect of risk on stock prices, relative to the exponential case.

The non-linear comparative statics induced by hyperbolic discounting should also have interesting implications for macroeconomic policy. Gong, Smith, Turnovsky, and Zou (2006) incorporate hyperbolic discounting into a model of fiscal policy in a stochastic growing economy. In the presence of hyperbolic discounting taxes on the stochastic components of capital and wage income have magnified effects on growth rates and welfare, relative to the benchmark exponential model.

APPENDIX A
Derivation of Proposition 1.

The transversality condition is

$$\lim_{t\to\infty} Ebe^{-\beta t} W_t^{1-\gamma} = 0. \tag{A.1}$$

As in Merton (1969, 1971) feasibility $c_H > 0$ is necessary and sufficient for the TVC to be satisfied. If the TVC is satisfied then

$$e^{-\beta h} E_0(W_h/W_0)^{1-\gamma} < 1. \tag{A.2}$$

Therefore, since $\delta \leq 1$, it must also be true that

$$1 - (1-\delta)hc_H e^{-\beta h} E_0(W_h/W_0)^{1-\gamma} > 0 \tag{A.3}$$

Inequalities (A.2) and (A.3) will be important in the ensuing comparative statics.

For small h, $hc_H < 1$. Equation (7) in the text implies that for small h it must also be true that $\gamma > 1-\delta$ in order for $c_H > 0$. Given $\gamma > 1-\delta$ it then follows that, $\gamma + [(1-\gamma)c_H h - 1](1-\delta)e^{-\beta h} E_0(W_h/W_0)^{1-\gamma} > 0$ for sufficiently small h.

The first statement follows from comparing Equations (11) and (12) and using the fact that $1 > c_H h$.

The second statement follows from differentiating Equation (12):

$$\frac{\partial^2 c_H}{\partial {\sigma^2}^2} = (1-\gamma)\gamma \frac{(1-\delta)he^{-\beta h}}{2}\Omega, \tag{A.4}$$

where

$$\Omega = \frac{[(1-\delta)e^{-\beta h}E_0(W_h/W_0)^{1-\gamma} - 1]E_0(W_h/W_0)^{1-\gamma} h\frac{\partial c_H}{\partial \sigma^2} - (1-\gamma)(1-hc_H)[\frac{\partial c_H}{\partial \sigma^2} + \frac{\gamma}{2}]}{\{\gamma + [(1-\gamma)c_H h - 1](1-\delta)e^{-\beta h}E_0(W_h/W_0)^{1-\gamma}\}^2}. \tag{A.5}$$

Again, $1 > hc_H$ for small h. We have seen that the transversality condition implies that the first term in braces is negative.

Consider the two cases mentioned in the proposition. On the one hand, if $\gamma < 1$ then $\partial c_H/\partial\sigma^2 > 0$, so $\Omega < 0$. Therefore $\partial^2 c_H/\partial {\sigma^2}^2 < 0$. On the other hand, if $\gamma > 1$ then $\partial c_H/\partial \sigma^2 < 0$. If the last term in braces is positive then $\Omega > 0$. It can be shown that this expression is positive if and only if $1 > hc_H$. Thus if $\gamma < 1$ then $\partial^2 c_H/\partial {\sigma^2}^2 < 0$.

APPENDIX B
Derivation of Proposition 2

The derivation is similar to that in Smith (2001). First, since all dividends are consumed, it follows that $D_t = c_H^* W_t$, where c_H^* denotes the equilibrium value of the MPC. However $W_t = P_t$ because there is one share of stock and the riskless asset is in zero net supply. Therefore

$$D_t = c_H^* P_t. \tag{B.1}$$

Now evaluate the MPC in Equation (7) at $\lambda_t = 1$. This yields Equation (10), which we report here for convenience

$$c_H^* = \frac{|\theta + (1-\delta) c_H^* e^{-\theta h + (1-\gamma)[\mu - c_H^* - \gamma\sigma^2/2]h}| - (1-\gamma)|\mu - \gamma\sigma^2/2|}{\gamma}. \tag{B.2}$$

Apply Ito's lemma to the function $f(D_t)$:

$$\frac{dP_t}{P_t} = \left[\frac{f'(D_t)}{f(D_t)} + \frac{f''(D_t)}{f(D_t)}\right] \nu D_t dt + \frac{f'(D_t)}{f(D_t)} \sigma_D D_t dZ_t. \tag{B.3}$$

This implies that the mean and variance of capital gains are

$$\pi = \left[\frac{f'(D_t)D_t}{f(D_t)} + \frac{f''(D_t)D_t}{f(D_t)}\right] \nu \tag{B.4}$$

$$\sigma^2 = \left[\frac{f'(D_t)D_t}{f(D_t)}\right]^2 \sigma_D^2. \tag{B.5}$$

Consider the term $\mu - c_H^*$ in the exponential function in Equation (B.2): By definition $\mu = \pi + D_t/P_t$. In equilibrium, however, $D_t/P_t = c_H^*$. Therefore $\mu - c_H^* = \pi$. Using this fact along with Equations (B.4) and (B.5) yields

$$\begin{aligned} &1 - (1-\delta) e^{-\theta h + (1-\gamma)([\frac{f'(D_t)D_t}{f(D_t)} + \frac{f''(D_t)D_t}{f(D_t)}]\nu - \gamma[\frac{f'(D_t)D_t}{f(D_t)}]^2 \frac{\sigma_D^2}{2})h} \\ =& f\left[\theta - (1-\gamma)\left(\left[\frac{f'(D_t)D_t}{f(D_t)} + \frac{f''(D_t)D_t}{f(D_t)}\right]\nu - \gamma\left[\frac{f'(D_t)D_t}{f(D_t)}\right]^2 \frac{\sigma_D^2}{2}\right)\right] \end{aligned} \tag{B.6}$$

This is equation (15) in the text.

Conjecture that the equilibrium price is proportional to dividends:[10]

$$P_t = A_H D_t. \tag{B.7}$$

[10]We ignore bubble solutions.

It follows that $\pi = \nu$ and $\sigma^2 = \sigma_D^2$. Equation (B.6) then reduces to

$$1 - (1-\delta)e^{-\theta h + (1-\gamma)(\nu - \gamma\sigma_D^2)h} = A_H \left[\theta - (1-\gamma)\left(\nu - \gamma\frac{\sigma_D^2}{2}\right)\right] \quad \text{(B.8)}$$

Solving for yields

$$A_H = \frac{1 - (1-\delta)e^{-\theta h + (1-\gamma)(\nu - \gamma\sigma_D^2)h}}{\theta - (1-\gamma)\left(\nu - \gamma\frac{\sigma_D^2}{2}\right)}. \quad \text{(B.9)}$$

This determines the equilibrium pricing function. To find the equilibrium interest rate, note that since $\lambda_t = 1$ in equilibrium, then $\mu - r = \gamma\sigma^2$. However, we have seen that $\mu = \pi + D_t/P_t, \pi = \nu$, and $\sigma^2 = \sigma_D^2$. Therefore $r = \nu + 1/A_H - \gamma\sigma_D^2$.

APPENDIX C
Derivation of Proposition 4

To simplify notation, define $x = \theta - (1-\gamma)(\nu - \sigma_D^2/2)$. The price-dividend ratio in Equation (B.9), or in Equation (17) in the text, can then be written as

$$A_H = \frac{1 - (1-\delta)e^{-xh}}{x}. \quad \text{(C.1)}$$

Similarly, the price-dividend ratio for the exponential model [Equation (19) in the text] is simply

$$A_M = \frac{1}{x}. \quad \text{(C.2)}$$

Equations (20) and (21) can now be expressed as $\partial A_m/\partial\sigma_D^2 = -\frac{x_{\sigma^2}}{x^2}$ and $\partial A_m^2/\partial{\sigma_D^2}^2 = x_{\sigma^2}^2/x^3$, where $x_{\sigma^2} = \partial x/\partial\sigma_D^2 = \gamma(1-\gamma)/2$.

Now consider the marginal effect of risk on the asset price:

$$\frac{\partial A_H}{\partial\sigma_D^2} = \frac{\partial A_M}{\partial\sigma_D^2}[1 - (1-\delta)e^{-xh}(1+xh)] \quad \text{(C.3)}$$

To sign this expression, substitute A_H in Equation (B.9) into inequality (A.3). This implies that the expression in brackets in Equation (C.3) is unambiguously positive. Therefore, $\partial A_H/\partial\sigma_D^2 >=< 0$ as $\gamma >=< 1$. Equation (C.3) also implies that $\left|\frac{\partial A_H}{\partial\sigma_D^2}\right| < \left|\frac{\partial A_M}{\partial\sigma_D^2}\right|$.

With a bit of tedious algebra it can be shown that

$$\frac{\partial A_H^2}{\partial{\sigma_D^2}^2} = \frac{\partial A_M^2}{\partial{\sigma_D^2}^2}\left[1 - \frac{1}{2}(1-\delta)e^{-xh}(1 + xh + x^2h^2)\right]. \quad \text{(C.4)}$$

To sign this expression, recall that $1 > (1-\delta)e^{-xh}(1+xh)$. Now consider the quadratic expression in Equation (C.4). It is straightforward to show that[11]

$$1 + xh > \frac{1 + xh + x^2h^2}{2}. \tag{C.5}$$

It follows that the expression in brackets is positive for small h. Since the exponential price function is convex in risk, the hyperbolic price function must also. That is, $\frac{\partial A_H^2}{\partial \sigma_D^2{}^2} > 0$.

REFERENCES

Ainslie, G. and N. Haslam, 1992, Hyperbolic Discounting, in G. Loewenstein and J. Elster, eds., *Choice Over Time*, (Russell Sage Foundation, New York) 57-92.

Bakshi, G. and Z. Chen, 1996, The Spirit of Capitalism and Stock Market Prices. *American Economic Review* **86**, 133-157.

Barro, R., 1999, Ramsey Meets Laibson in the Neoclassical Growth Model. *Quarterly Journal of Economics* **114**, 1125-1152.

Bernheim, D., D. Ray, and S. Yeltekin, 1999, Self-Control, Saving, and the Low Asset Trap, Working Paper, Stanford University.

Duffie, D. and L. Epstein, 1993a, Asset Pricing with Stochastic Differential Utility. *Review of Financial Studies* **5**, 411-436

Duffie, D. and L. Epstein, 1993b, Stochastic Differential Utility. *Econometrica* **60**, 353-394.

Epstein, L., 1987, Risk Aversion and Asset Prices. *Journal of Monetary Economics* **22**, 179-192.

Epstein, L. and S. Zin, 1989, Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica* **57**, 937-969.

Epstein, L. and S. Zin, 1991, Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis. *Journal of Political Economy* **99**, 263-286.

Gong, L., W. Smith, and H. Zou, 2006, Consumption and Risk with Hyperbolic Discounting, forthcoming, Economics Letters.

Gong, L., W. Smith, S. Turnovsky, and H. Zou., 2006, Fiscal Policies in a Stochastic Growth Model with Hyperbolic Discounting, Working Paper, Beijing University.

Gong, L. and H. Zou, 2002, Direct Preferences for Wealth, the Risk Premium Puzzle, Growth, and Policy Effectiveness. *Journal of Economic Dynamics and Control* **26**, 243-270.

Gong, L. and H. Zou, 2002, Money, Social Status, and Capital Accumulation in a Cash-in-Advance Model. *Journal of Money Credit and Banking*, **33**, 284-293.

[11]Proof: If this inequality holds then $1+xh > x^2h^2$. When plotted as a function of h the left hand side of this inequality has a positive vertical intercept of 1 and a positive slope of x. The right hand side is equal to zero when $h = 0$ and at the rate $2x^2h$. Therefore for sufficiently small h the left hand side is greater than the right hand side.

Gootzeit, M, J. Schneider, and W. Smith, 2002, Marshallian Recursive Preferences and Growth. *Journal of Economic Behavior and Organization* **49**, 381-404.

Gul, F. and W. Pesendorfer, 2001, Temptation and Self-Control. *Econometrica* **69**, 1403-1435.

Gul, F. and W. Pesendorfer, 2002, Self-Control and the Theory of Consumption, Working Paper, Princeton University.

Harris, C. and D. Laibson, 2001a, Dynamic Choices of Hyperbolic Consumers. *Econometrica* **69**, 935-957.

Harris, C. and D. Laibson, 2001b, Instantaneous Gratification, Working Paper, Harvard University.

Harris, C. and D. Laibson, 2003, Hyperbolic Discounting and Consumption, in M. Dewatripont, L. Hansen, and S. Turnovsky, eds., Advances in Economics and Econometrics: Theory and Applications, Eight World Congress, Vol. I, (Cambridge University Press, Cambridge).

Krusell, P. and A. Smith, 2001, Consumption and Savings Decisions with Quasi-Geometric Discounting, Working Paper, Carnegie Mellon University.

Laibson, D., 1994, Self Control and Savings, Ph.D. Dissertation, Massachusetts Institute of Technology.

Laibson, D., 1997a, Golden Eggs and Hyperbolic Discounting. *Quarterly Journal of Economics* **112**, 443-478.

Laibson, D., 1997b, Hyperbolic Discount Functions and Time Preference Heterogeneity, Working Paper, Harvard University.

Lucas, R., 1978, Asset Prices in an Exchange Economy. *Econometrica* **46**, 1429-1445.

Luttmer, E. and T. Mariotti, 2002, Subjective Discounting in an Exchange Economy. *Journal of Political Economy* **111**, 959-989.

Marshall, A., 1979, *Principles of Economics*, 8th edition (MacMillan, London).

Merton, R.C., 1969, Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case. *Review of Economics and Statistics* **51**, 247-257.

Merton, R.C., 1971, Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory* **3**, 373-413.

Morris, S. and A. Postlewaite, 1997, Observational Implications on Nonexponential Discounting, Working Paper.

O'Donoghue, T. and M. Rabin, 1999, Doing It Now or Later. *American Economic Review* **89**, 103-124.

Palacios-Huerta, I., 2002, Consumption and Portfolio Rules Under Hyperbolic Discounting, Working Paper, Brown University.

Smith, W., 1999, Risk, the Spirit of Capitalism, and Long Term Growth: the Implications of a Preference for Capital. *Journal of Macroeconomics* **21**, 241-261.

Smith, W., 2001, How Does the Spirit of Capitalism Affect Stock Market Prices. *Review of Financial Studies* **14**, 1215-1232.

Svensson, L., 1989, Portfolio Choice with Non-expected Utility in Continuous Time. *Economics Letters* **30**, 313-317.

Turnovsky, S., 2000a, Methods of Macroeconomic Dynamics., 2nd ed., MIT Press.

Turnovsky, S., 2000b, Government Policy in a Stochastic Growth Model with Elastic Labor Supply. *Journal of Public Economic Theory* **2**, 389-433.

Turnovsky, S. 2000c, Fiscal Policy, Elastic Labor Supply, and Endogenous Growth. *Journal of Monetary Economics* **45**, 185-210.

Weber, M., 1958, The Protestant Ethic and the Spirit of Capitalism (Charles Scribner's Sons, New York).

Weil, P., 1989, Non-expected Utility in Macroeconomics. *Quarterly Journal of Economics* **105**, 29-42.

Zou, H., 1994, The 'Spirit of Capitalism' and Long Run Growth. *European Journal of Political Economy* **10**, 279-293.

Zou, H., 1995, The Spirit of Capitalism and Savings Behavior. *Journal of Economic Behavior and Organization* **28**, 131-143.