ANNALS OF ECONOMICS AND FINANCE **12-2**, 217–231 (2011)

A Robust General Equilibrium Stochastic Volatility Model with Recursive Preference Investors

Weidong Xu*

School of Management, Zhejiang University, Hangzhou 310058, China
E-mail: weid.xu@gmail.com

Hongyi Li

Business Administration Faculty
The Chinese University of Hong Kong, Shatin, N.T., Hong Kong
E-mail: hongyi@baf.msmail.cuhk.edu.hk

and

Chongfeng Wu

Financial Engineering Research Center, Shanghai Jiaotong University, Shanghai 200052, China
E-mail: cfwu@sjtu.edu.cn

This paper investigates the implications of model uncertainty for the equity premium in a stochastic volatility model. We consider a general equilibrium setting with one representative agent who has a stochastic differential utility. The results show that the equilibrium equity premium consists of a market risk premium, a stochastic volatility risk premium and an uncertainty aversion premium. Further, the robustness can increase the equilibrium equity premium and drive down the equilibrium risk-free rate.

Key Words: General equilibrium; Robust control; Stochastic volatility model; Equity premium.
JEL Classification Numbers: C61, D51, D81.

* The authors would like to acknowledge the financial support of National Natural Science Foundation of China (No. 71101125 and 70831004), Distinguished Young Scholar of Zhejiang University, and QianJiang Talent Plan of Zhejiang Province.

1529-7373/2011

1. INTRODUCTION

This paper studies the implications of the presence of any uncertainty about the growth rate process for the asset pricing by adopting a general equilibrium setting with a robust agent who has a stochastic differential utility. The paradox discussed in Ellsberg (1961) illustrated that the behavior of investors under uncertainty aversion and standard risk aversion is inherently different. Therefore, it is very important to distinguish between uncertainty aversion and risk aversion from both economic and behavioral point of views. Recently, model uncertainty, or fear of model misspecification, has been widely studied in portfolio choice and asset pricing in case of both multiple priors and robust control, e.g., Epstein and Wang (1994), Epstein and Chen (2002), Anderson, Hansen, and Sargent (2003), Uppal and Wang (2003), Maenhout (2004), and Liu, Pan, and Wang (2005), Miao (2004, 2009), Garlappi, Uppal, and Wang (2007), Liu (2010), among others. Wang and Zou (2011) and Zou, Gong, and Zeng (2011) investigated the monetary growth rate with inflation aversion.

Following Anderson, Hansen, and Sargent (2003), the robust control framework is adopted in this paper to account for the imprecise knowledge about the probability distribution with respect to the fundamental risks in the economy. Without considering stochastic volatility, in a pure-exchange economy with one representative agent Maenhout (2004) studied the implications of model uncertainty for equity premium in a diffusion model, while Liu, Pan, and Wang (2005) solved the equilibrium asset prices in a jump diffusion model by allowing for model uncertainty with respect to rare events. Xu, Wu, and Li (2010) investigated the implications of ambiguity for the equity premium in a stochastic volatility model by adopting a general equilibrium setting. In order to obtain the analytical results with stochastic volatility risk, the representative agent is assumed to have a simple log utility in Xu, Wu, and Li (2010). Given that the risk aversion parameter γ usually does not take the value of 1, we extend the analysis in Xu, Wu, and Li (2010) to the case of a representative agent with stochastic differential utility. In particular, the agent also worries about the model uncertainty. The results show that in equilibrium the equity premium consists of three parts: market risk premium, stochastic volatility risk premium, and uncertainty aversion premium. Further, the equilibrium risk-free rate depends on time preference, intertemporal substitution and growth, and precautionary savings in response to consumption uncertainty, and the robustness drives down the equilibrium risk-free rate through the precautionary savings channel.

The rest of the paper is organized as follows. Section 2.1 sets up the framework of standard control for a representative agent with recursive preference in a pure-exchange economy. Alternative models with respect to the reference model are introduced in Section 2.2. Section 2.3 solves the optimal portfolio and consumption problem for an investor who exhibits aversion to both risk and uncertainty. The equilibrium results are derived in Section 2.4. Finally, the concluding remarks are given in Section 3.

2. THE MODEL

We consider a continuous time pure-exchange economy with a representative agent who has utility over consumption streams (Lucas, 1978). In our model, the agent is assumed to be uncertain about the growth rate of output. Given the agent with a recursive preference, the near-explicit closed-form solutions of equilibrium risk-free rate and equity premium can be obtained. The treatment and notations of model uncertainty in this section will follow those of Anderson, Hansen, and Sargent (2003) and Maenhout (2004).

2.1. The economy

Similar to Xu, Wu, and Li (2010), we study a pure-exchange economy in which the representative agent is endowed with shares in a production technology that generates a dividend flow D_t. The economy is populated by the agent who maximizes his/her expected lifetime utility and has access to two financial assets: one riskless, paying an instantaneous rate r_t, and the other risky (equities), paying the dividend process D_t.

Consider a probability space $(\Omega, \mathscr{F}, \mathcal{P})$ endowed with a standard complete filtration $(\mathscr{F}_t | t \geq 0)$. We assume that D_t is a Markov process in $\mathbb{R}$ with respect to the filtration $(\mathscr{F}_t)$ in some state space $\mathfrak{D} \subset \mathbb{R}$ solving the following stochastic differential equations

$$dD_t = \mu_{_D} D_t dt + \sqrt{v_t} D_t (\rho dB_t + \sqrt{1-\rho^2} dZ_t), \tag{1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma_v \sqrt{v_t} dB_t, \tag{2}$$

where B_t and Z_t are standard independent Brownian motions. This endowment flow model is the standard stochastic volatility model of Heston (1993) with constant mean growth rate $\mu_{_D} \geq 0$ and local variance v_t. Here v_t is a square-root mean reverting process with long-run mean θ, speed of adjustment κ, and variation coefficient σ_v of the diffusion volatility v_t. The parameters κ, θ, and σ_v are assumed to be nonnegative and satisfy $2\kappa\theta \geq \sigma_v^2$. We assume that the price S_t of the risky asset which is the claim to the dividend stream satisfies an Itô process

$$dS_t = \Big(S_t \mu_{_S} - \frac{S_t}{D_t}\Big) dt + \sigma_{_S} S_t (\rho dB_t + \sqrt{1-\rho^2} dZ_t), \tag{3}$$

where the coefficients μ_S and σ_S are determined in the general equilibrium. The conjecture for risky asset price process (3) implies that the cumulative return on the risky asset is simply

$$\frac{dS_t}{S_t} + \frac{D_t}{S_t}dt = \mu_S dt + \sigma_S(\rho dB_t + \sqrt{1-\rho^2}dZ_t). \tag{4}$$

Denoting the risk-free return by r, and the fraction of wealth allocated to the risky asset at time t by π_t, the state equation for wealth are

$$dW_t = [W_t(r + \pi_t(\mu_S - r))]dt - C_t dt + \pi_t \sigma_S W_t(\rho dB_t + \sqrt{1-\rho^2}dZ_t). \tag{5}$$

Both control variables $\{\pi_t, C_t\}$ are nonanticipating and suitably adapted to the σ-algebra generated by the underlying Brownian motions.

We assume that the agent has continuous time Epstein and Zin (1989) recursive preferences over consumption facing the endowment process D_t defined in equations (1) and (2). Following the terminology of Duffie and Epstein (1992a), the agent's utility is determined by

$$J_t = \int_t^\infty f(C_s, J_s)ds \tag{6}$$

where $f(C_t, J_t)$ is a normalized aggregator of current consumption and continuation value that takes the form

$$f(C, J) = \frac{\delta}{1-\frac{1}{\psi}}(1-\gamma)J\left\{\left\{\frac{C}{[(1-\gamma)J]^{\frac{1}{1-\gamma}}}\right\}^{1-\frac{1}{\psi}} - 1\right\}. \tag{7}$$

Here δ is the rate of time preference, γ is the agent's relative risk aversion, and ψ is the intertemporal elasticity of substitution (IES). An important special case of this aggregator is $\gamma = \frac{1}{\psi}$, in which case the normalized aggregator (7) reduces to the standard additive power utility function. The log utility function can be obtained when $\gamma = \psi = 1$. If $\psi = 1$, the aggregator $f(C, J)$ takes the following form, see Campbell et al. (2004),

$$f(C, J) = \delta(1-\gamma)J\Big[\log(C) - \frac{1}{1-\gamma}\log((1-\gamma)J)\Big]. \tag{8}$$

The case $\psi = 1$ is important because it allows an exact solution for the equilibrium risk-free interest rate and equity premium.

For a given probability measure $\mathcal{P}$, the representative agent's utility is then given by

$$E_t\Big[\int_t^\infty f(C_s, J_s)ds\Big]. \tag{9}$$

In the absence of model uncertainty, the Hamilton-Jacobi-Bellman (HJB) equation for the value function J_t can be obtained following Duffie and Epstein (1992b)

$$\sup_{\{C_t,\pi_t\}} \{f(C_t, J_t) + \mathcal{A}J(W_t, v_t)\}, \tag{10}$$

where

$$\begin{aligned}\mathcal{A}J(W_t, v_t) = [W_t(r + \pi(\mu_S - r)) - C_t]J_W + [\kappa(\theta - v_t)]J_v + \frac{1}{2}\pi^2\sigma_S^2 W_t^2 J_{WW} \\ + \frac{1}{2}\sigma_v^2 v_t J_{vv} + \rho\pi\sigma_S W_t \sigma_v \sqrt{v_t} J_{Wv}.\end{aligned} \tag{11}$$

Note that J_{Wv} is the second partial derivative with respective to v and W; J_W and J_{WW} are its first and second derivatives with respect to W; and finally J_v and J_{vv} are its first and second derivatives with respect to v. J_t denotes the value of J at time t.

2.2. Alternative models

In this section, we assume that the representative agent does not fully trust the benchmark or reference economy's dynamic model described by equations (1) and (2). Therefore, the model uncertainty leads the agent to consider a set of alternative models in his/her decision process. We deviate from the standard approach by considering a representative agent who, in addition to being risk averse, exhibits uncertainty aversion in the sense of Knight (1921) and Ellsberg (1961). An alternative model is defined by its probability measure. Let $\mathcal{P}$ be the probability measure associated with the reference model described by equations (1) and (2). The alternative model is then defined by a probability measure $\mathcal{P}(\xi)$, where $\xi_T = d\mathcal{P}(\xi)/d\mathcal{P}$ is its Radon-Nikodym derivative (likelihood ratio) with respect to $\mathcal{P}$. It is useful to specify models through their Radon-Nikodym derivative since this permits a convenient definition of the set of models that are statistically close to the reference model.

According to the Girsanov theorem, the Radon-Nikodym derivative of $\xi_T = d\mathcal{P}(\xi)/d\mathcal{P}$ is defined by

$$\xi_t = \exp\left\{\int_0^t (-h_s dB_s - g_s dZ_s) - \frac{1}{2}\int_0^t (h_s^2 + g_s^2)ds\right\}, \tag{12}$$

where h_t and g_t are measurable functions such that $\int_0^\infty h_s^2 < \infty$ a.s. and $\int_0^\infty h_s^2 < \infty$ a.s., respectively; and ξ is a $\mathcal{P}$ martingale with $\xi_0 = 1$. Furthermore, by Girsanov's theorem,

$$dB_t^\xi = h_t dt + dB_t, \tag{13}$$

$$dZ_t^\xi = g_t dt + dZ_t \tag{14}$$

are Brownian motions under $\mathcal{P}(\xi)$, and the alternative model is defined by equations (15) and (16) below

$$\frac{dS_t}{S_t} + \frac{D_t}{S_t}dt = (\mu_S + \rho\sigma_S h_t + \sqrt{1-\rho^2}\sigma_S g_t)dt + \sigma_S(\rho dB_t + \sqrt{1-\rho^2}dZ_t), \tag{15}$$

$$dv_t = [\kappa(\theta - v_t) + \sigma_v\sqrt{v_t}g_t]dt + \sigma_v\sqrt{v_t}dB_t. \tag{16}$$

In the current diffusion setting, by Girsanov's theorem we can see that the alternative model focuses on the subclass of alternative models that only differ in terms of the drift functions. Therefore the general model uncertainty is reduced to uncertainty about the drift function of the state variable. The state equation for wealth now becomes

$$\begin{aligned} dW_t = {} & [W_t(r + \pi_t(\mu_S - r))]dt + W_t\pi_t\sigma_S(\rho h_t + \sqrt{1-\rho^2}g_t)dt - C_t dt \\ & + W_t\pi_t\sigma_S(\rho dB_t + \sqrt{1-\rho^2}dZ_t). \end{aligned} \tag{17}$$

Under model uncertainty, the representative agent's utility function (9) can be written as

$$J_t = \min_{\mathcal{P}(\xi)} E_t^{\xi}\Big[\int_t^{\infty} f(C_s, J_s)ds\Big]. \tag{18}$$

In this case, the agent chooses optimally a distortion to the reference model [equations (1) and (2)] in a way that makes his decisions robust to statistically small model misspecifications. Following Duffie and Epstein (1992b) and Anderson, Hansen, and Sargent (2003), the Hamilton-Jacobi-Bellman equation for the value function J is given by

$$\begin{aligned} 0 = \sup_{\{C_t,\pi_t\}} \inf_{\{h_t,g_t\}} \Big\{ & f(C_t, J_t) + \mathcal{A}J(W,v) + (W_t\pi_t\sigma_S\rho h_t + W_t\pi_t\sigma_S\sqrt{1-\rho^2}g_t)J_W \\ & + \sigma_v\sqrt{v_t}h_t J_v + \frac{1}{2\beta}(h_t^2 + g_t^2)\Big\}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \mathcal{A}J(W_t, v_t) = {} & [W_t(r + \pi(\mu_S - r)) - C_t]J_W + [\kappa(\theta - v_t)]J_v + \frac{1}{2}\pi^2\sigma_S^2 W_t^2 J_{WW} \\ & + \frac{1}{2}\sigma_v^2 v_t J_{vv} + \rho\pi\sigma_S W_t\sigma_v\sqrt{v_t}J_{Wv}. \end{aligned} \tag{20}$$

The parameter β measures the magnitude of the desired robustness. When β is set to 0, any alternative model $\mathcal{P}(\xi)$ that deviates from the reference model will be penalized heavily and the agent has complete confidence in the reference model, in which case equation (18) reduces to the standard expected utility in equation (9). In other words, an agent with higher β exhibits higher aversion to model uncertainty and has more desire for robustness. For more discussions about the model uncertainty under continuous time recursive preference expected utility of the form (18), see Maenhout (2004) and Liu, Pan, and Wang (2005). Compared with the standard HJB equation (10) for geometric Brownian motion with stochastic volatility, the HJB equation in (19) has two important modifications. First, the risk associated with the diffusion shock and stochastic volatility components are evaluated at all possible alternative models indexed by (h_t, g_t), reflecting the agent's precaution against substantial uncertainty about the expected growth drift process and stochastic volatility drift process. Second, it incorporates an additional term, $\frac{1}{2\beta}(h_t^2 + g_t^2)$, penalizing the choice of the alternative model by its distance from the reference model.

2.3. The optimal consumption and portfolio choice

For the following optimization problem

$$\inf_{\{h_t, g_t\}} \Big\{ f(C_t, J_t) + \mathcal{A}J(W, v) + (W_t \pi_t \sigma_S \rho h_t + W_t \pi_t \sigma_S \sqrt{1-\rho^2} g_t) J_W + \sigma_v \sqrt{v_t} h_t J_v + \frac{1}{2\beta}(h_t^2 + g_t^2) \Big\}, \tag{21}$$

the first order conditions with respect to h_t and g_t are

$$\begin{pmatrix} h_t \\ g_t \end{pmatrix} = -\beta \begin{pmatrix} W_t \pi_t \sigma_S \rho J_W + \sigma_v \sqrt{v_t} J_v \\ W_t \pi_t \sigma_S \sqrt{1-\rho^2} J_W \end{pmatrix}. \tag{22}$$

Substituting the first order conditions (22) back into the HJB equation (19), we obtain

$$0 = \sup_{\{C_t, \pi_t\}} \Big\{ f(C_t, J_t) + \mathcal{A}J(W, v) - \frac{\beta}{2}\Big(W_t^2 \pi_t^2 \sigma_S^2 J_W^2 + 2 W_t \pi_t \sigma_S \rho \sigma_v \sqrt{v_t} J_W J_v + \sigma_v^2 v_t J_v^2 \Big) \Big\}. \tag{23}$$

By the first order conditions, the optimal consumption and portfolio choice are given by

$$C_t = J_W^{-\psi} [(1-\gamma) J]^{\frac{1-\gamma\psi}{1-\gamma}} \delta^{\psi}, \ if \ \psi \neq 1, \tag{24}$$

$$C_t = \frac{J}{J_W}(1-\gamma)\delta, \ if \ \psi = 1, \tag{25}$$

and

$$\pi_t = \frac{1}{(1 - \frac{\beta J_W^2}{J_{WW}})} \frac{-J_W}{W_t J_{WW}} \frac{\mu_S - r}{\sigma_S^2} + \frac{1}{(1 - \frac{\beta J_W^2}{J_{WW}})} \frac{\rho \sigma_v \sqrt{v_t}}{J_{WW} W_t \sigma_S} (\beta J_W J_v - J_{Wv}). \tag{26}$$

The first element of the optimal portfolio choice corresponds to a variant of the usual myopic demand for the risky asset, where

$$\frac{1}{1 - \frac{\beta J_W^2}{J_{WW}}} \tag{27}$$

is the adjustment coefficient of the usual portfolio weight for risky asset in Merton (1971). Therefore robustness amounts to an increase in the effective risk aversion, and pushes down the demand schedule for the risky asset. The second element is the volatility hedging demand for the risky asset, which is directly related to the correlation coefficient ρ between instantaneous returns on the risky asset and changes in the volatility v. When this correlation is nonzero, the investor can hedge his/her expected utility against the shifts in v by taking a position in the risky asset.

The closed form solution for the above model can be obtained for the representative agent with simple log utility, see, Xu, Wu, and Li (2010). To explicitly solve the model with stochastic differential utility, following Maenhout (2004) we replace the robust preference parameter β by a state-dependent version of $\hat{\beta}(W, v)$ which scales β by a value function:

$$\hat{\beta}(W, v) = \frac{\beta}{(1-\gamma) J(W, v)}. \tag{28}$$

Then the HJB equation becomes

$$\begin{aligned} 0 = \sup_{\{C_t, \pi_t\}} \Big\{ f(C_t, J_t) + \mathcal{A} J(W, v) - \Big(\frac{\beta}{2(1-\gamma)} \frac{W_t^2 \pi_t^2 \sigma_S^2 J_W^2}{J} \\ + \frac{\beta}{(1-\gamma)} \frac{W_t \pi_t \sigma_S \rho \sigma_v \sqrt{v_t} J_W J_v}{J} + \frac{\beta}{2(1-\gamma)} \frac{\sigma_v^2 v_t J_v^2}{J} \Big) \Big\}. \end{aligned} \tag{29}$$

For different IES ($\psi = 1$ or $\psi \neq 1$), this equation results in different forms of partial differential equations by using different normalized aggregators (7) or (8). According to different values of ψ, the optimal consumption and portfolio choice are obtained in Propositions 1 and 2, respectively.

Proposition 2.1. *When $\psi = 1$ and $\gamma \neq 1$, the solution to HJB equation (29) is given by*

$$J(W_t, v_t) = I(v_t)^{1-\gamma} \frac{W_t^{1-\gamma}}{1-\gamma} \tag{30}$$

where $I(v_t) = \exp\{A_0 + B_0 v_t\}$, and A_0 and B_0 are functions of the primitive parameters of the model describing investment opportunities and preference and satisfy the following recursive equation

$$\begin{aligned}
0 = {}& \delta\Big(\log(\delta) - A_0 - B_0 v_t\Big) + \Big(r + \frac{(\mu_S - r)^2}{(\gamma+\beta)\sigma_S^2} + \frac{1-\gamma-\beta}{\gamma+\beta}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}(\mu_S - r)B_0 - \delta\Big) \\
& - \frac{1}{2}\gamma\sigma_S^2\Big(\frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} + \frac{1-\gamma-\beta}{\gamma+\beta}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_0\Big)^2 + B_0\kappa(\theta - v_t) + \frac{1}{2}\sigma_v^2 v_t(1-\gamma)B_0^2 \\
& + \rho\sigma_S\sigma_v\sqrt{v_t}(1-\gamma)B_0\Big(\frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} + \frac{1-\gamma-\beta}{\gamma+\beta}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_0\Big) - \frac{\beta}{2}\sigma_v^2 v_t B_0^2 \\
& - \frac{\beta}{2}\sigma_S^2\Big(\frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} + \frac{1-\gamma-\beta}{\gamma+\beta}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_0\Big)^2 \\
& - \beta\sigma_S\rho\sigma_v\sqrt{v_t}B_0\Big(\frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} + \frac{1-\gamma-\beta}{\gamma+\beta}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_0\Big). \qquad (31)
\end{aligned}$$

The optimal consumption and portfolio choice are given by

$$C_t = \delta W_t, \qquad (32)$$

and

$$\pi_t = \frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} + \frac{1-\gamma-\beta}{\gamma+\beta}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_0. \qquad (33)$$

Specially, when $\psi = 1$ and $\gamma = 1$, the solution to HJB equation (29) is given by

$$J(W_t, v_t) = \log(W_t) + \hat{A}_0 + \hat{B}_0 v_t \qquad (34)$$

where $\hat{A}_0$ and $\hat{B}_0$ satisfy the recursive equation given in (31) with $\gamma = 1$. The optimal consumption is given by equation (32) and the optimal portfolio choice is given by equation (33) with $\gamma = 1$.

Proof. Substitution of equations (25), (26), and (30) into the HJB equation (29) leads, after some simplification, to equation (31). The optimal consumption and portfolio choice can be obtained by using equations (30) and (31) from the first order conditions (25) and (26). For the special case of $\psi = 1$ and $\gamma = 1$, we notice that the robust preference parameter $\hat{\beta}(W, v)$ reduces to constant beta

$$\lim_{\gamma\to 1}\hat{\beta}(W, v) = \lim_{\gamma\to 1}\frac{\beta}{W_t^{1-r}\exp\{(1-\gamma)(A_0 + B_0 v_t)\}} = \beta. \qquad (35)$$

Then we can show that the Bellman equation (29) can be solved by equation (34) and the parameters $\hat{A}_0$ and $\hat{B}_0$ satisfy the recursive equation given in (31) with $\gamma = 1$.

In a more general case of $\psi \neq 1$, there is no exact analytical solution to the Bellman equation (29). However, we can use a log-linear approximate method introduced in Campbell et al. (2004) to find an approximate analytical solution. ▮

PROPOSITION 2.2. *When $\psi \neq 1$, the solution to HJB equation (29) is given by*

$$J(W_t, v_t) = H(v_t)^{-\frac{1-\gamma}{1-\psi}} \frac{W_t^{1-\gamma}}{1-\gamma} \tag{36}$$

where $H(v_t) = \exp\{A_1 + B_1 v_t\}$, and A_1 and B_1 are functions of the primitive parameters of the model describing investment opportunities and preference and satisfy the following recursive equation

$$\begin{aligned}
0 = {} & \frac{1}{1-\frac{1}{\psi}}\Big(h_0 + h_1(\psi\log(\beta) - h_t) - \beta\Big) \\
& + \Big(r + \frac{(\mu_S - r)^2}{(\gamma+\beta)\sigma_S^2} - \frac{1-\gamma-\beta}{(1-\psi)(\gamma+\beta)}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}(\mu_S - r)B_1 - h_0 - h_1(\psi\log(\beta) - h_t)\Big) \\
& - \frac{1}{2}\Big(\frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} - \frac{1-\gamma-\beta}{(1-\psi)(\gamma+\beta)}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_1\Big)^2\sigma_S^2\gamma \\
& - \frac{1}{1-\psi}B_1\kappa(\theta - v_t) + \frac{1}{2}\frac{1-\gamma}{(1-\psi)^2}\sigma_v^2 v_t B_1^2 \\
& - \rho\sigma_S\sigma_v\sqrt{v_t}\frac{1-\gamma}{1-\psi}B_1\Big(\frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} - \frac{1-\gamma-\beta}{(1-\psi)(\gamma+\beta)}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_1\Big) \\
& - \frac{\beta}{2}\sigma_S^2\Big(\frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} - \frac{1-\gamma-\beta}{(1-\psi)(\gamma+\beta)}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_1\Big)^2 \\
& + \frac{\beta}{1-\psi}\rho\sigma_S\sigma_v\sqrt{v_t}B_1\Big(\frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} - \frac{1-\gamma-\beta}{(1-\psi)(\gamma+\beta)}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_1\Big) \\
& - \frac{\beta}{2}\frac{1}{(1-\psi)^2}\sigma_v^2 v_t B_1^2.
\end{aligned} \tag{37}$$

By the log-*linear approximate method described in Campbell et al. (2004), the parameters h_0, h_1, and h_t are defined by $h_1 = \exp\{E[c_t - w_t]\}$, $h_0 = h_1(1 - \log(h_1))$, and $h_t = \log(\exp\{A_1 + B_1 v_t\})$, respectively, where $c_t = \log(C_t)$, and $w_t = \log(W_t)$.*

The optimal consumption and portfolio choice are given by

$$C_t = \exp\{-A_1 - B_1 v_t\} W_t \delta^{\psi}, \tag{38}$$

and

$$\pi_t = \frac{\mu_S - r}{(\gamma+\beta)\sigma_S^2} - \frac{1-\gamma-\beta}{(1-\psi)(\gamma+\beta)}\frac{\rho\sigma_v\sqrt{v_t}}{\sigma_S}B_1. \tag{39}$$

Proof. The optimal consumption (38) and portfolio choice (39) can be directly obtained by substituting equation (36) into the first order conditions (24) and (26) after some simplification. At the same time, substitution of equation (36) into the Bellman equation (29) gives, after some simplification, the following ordinary differential equation:

$$\begin{aligned}0 &= \frac{1}{1-\frac{1}{\psi}}(H(v_t)^{-1}\beta^{\psi}-\beta)+\Big(r+\pi_t(\mu_S-r)-H(v_t)^{-1}\beta^{\psi}\Big)-\frac{1}{2}\pi_t^2\sigma_S^2\gamma\\ &\quad -\frac{1}{1-\psi}B_1\kappa(\theta-v_t)+\frac{1}{2}\frac{1-\gamma}{(1-\psi)^2}\sigma_v^2 v_t B_1^2-\rho\sigma_S\sigma_v\sqrt{v_t}\frac{1-\gamma}{1-\psi}B_1\\ &\quad -\frac{\beta}{2}\sigma_S^2\pi_t^2+\frac{\beta}{1-\psi}\rho\sigma_S\sigma_v\sqrt{v_t}B_1\pi_t-\frac{\beta}{2}\frac{1}{(1-\psi)^2}\sigma_v^2 v_t B_1^2. \qquad (40)\end{aligned}$$

This ordinary differential equation does not have an exact analytical solution, unless $\psi = 1$. We notice that the first order condition for consumption (38) gives

$$\frac{C_t}{W_t} = H(v_t)^{-1}\delta^{\psi}. \qquad (41)$$

Following Campbell et al. (2004), we can now use the unconditional mean of the log consumption-wealth ratio to approximate $H(v_t)^{-1}\beta^{\psi}$:

$$H(v_t)^{-1}\delta^{\psi} = \exp\{c_t - w_t\} \approx h_0 + h_1(\psi\log(\delta) - h_t) \qquad (42)$$

where $c_t = \log(C_t)$, $w_t = \log(W_t)$, $h_t = \log(H(v_t))$, and

$$h_1 = \exp\{E[c_t - w_t]\}, \qquad (43)$$

$$h_0 = h_1(1 - \log(h_1)). \qquad (44)$$

Now substitution of approximation (42) and optimal portfolio choice (39) into equation (40) gives, after some simplification, the recursive equation (37). ∎

2.4. Market equilibrium

In a robust equilibrium model, the representative agent invests all his/her wealth in the stock market ($\pi_t = 1$) and consumes the aggregate endowment $C_t = D_t$ at any time $t \le T$. Given the closed-form solutions for equilibrium consumption-investment decisions, the equilibrium risk-free rate and the equity premium are obtained by the following propositions[1]. According to the different values of ψ, the equilibrium risk-free rate and the equity premium are obtained in Propositions 3 and 4, respectively.

[1]Without stochastic volatility considerations, Maenhout (2004) derived the equilibrium risk-free rate and the equity premium in a pure-exchange economy with a robust

PROPOSITION 2.3. *In equilibrium, for $\psi = 1$, the total (cum-dividend) equity premium is*

$$\mu_S - r = [\gamma + \beta + (\gamma + \beta - 1)\rho\sigma_v B_0]\sigma_{CS} \tag{45}$$

where $\sigma_{CS} = cov(\frac{dC}{C}, \frac{dS}{S})$, and the parameter B_0 is the solution to the following equations:

$$\delta \log(\delta) + \mu_D - \delta A_0 + B_0 k\theta = 0 \tag{46}$$

$$\frac{1}{2}\sigma_v^2(1 - \gamma - \beta)B_0^2 - (\delta + \kappa - \rho\sigma_v + \rho\sigma_v\gamma) + \beta\rho\sigma_v)B_0 - \frac{1}{2}\gamma - \frac{1}{2}\beta = 0 \tag{47}$$

The equilibrium risk-free rate is

$$r = \mu_D + \delta + \rho\sigma_v v_t B_0 - (\gamma + \beta)(v_t + \rho\sigma_v v_t B_0). \tag{48}$$

PROPOSITION 2.4. *In equilibrium, for $\psi \neq 1$, the total (cum-dividend) equity premium is*

$$\mu_s - r = \Big(\gamma + \beta + \frac{1 - \gamma - \beta}{1 - \psi}\rho\sigma_v B_1\Big)\sigma_{CS} \tag{49}$$

where $\sigma_{CS} = cov(\frac{dC}{C}, \frac{dS}{S})$, and the parameter B_1 is the solution to the following equations:

$$\frac{\psi}{\psi - 1}(h_0 + h_1\psi \log(\delta) - \delta) + \mu_D - \frac{\psi}{\psi - 1}k_1 A_1 - \frac{1}{1 - \psi}\kappa\theta B_1 = 0, \tag{50}$$

$$\frac{\sigma_v^2}{2(1 - \psi)^2}(1 - \gamma - \beta)B_1^2 + \frac{1}{1 - \psi}\Big(\kappa + \psi h_1 - \frac{1 - \gamma}{1 - \psi}\rho\sigma_v + \frac{\beta}{1 - \psi}\rho\sigma_v\Big)B_1 + \frac{1}{2}(1 - \gamma - \beta) = 0. \tag{51}$$

The equilibrium risk-free rate is

$$r = \mu_D + h_0 + h_1(\psi \log(\delta) - h_t) - (\gamma + \beta)v_t \left[1 + \frac{1 - \gamma - \beta}{(1 - \psi)(\gamma + \beta)}\rho\sigma_v B_1\right]. \tag{52}$$

---

representative agent that has Duffile-Epstein-Zin preference. By modelling rare events as jumps in the aggregate endowment, Liu, Pan, and Wang (2005) also obtained the equilibrium risk-free rate and the equity premium in a pure-exchange economy with a robust representative agent who has a Duffile-Epstein-Zin preference.

The proof of Proposition 3 is similar to that of Proposition 4. Therefore, in the following we only give the proof of Proposition 4.

Proof. The equilibrium condition in the goods market implies that $C_t = D_t = [h_0 + h_1(\psi \log(\delta) - h_t)]W_t$, where the last identity can be obtained by equations (41) and (42). Further the equilibrium condition in the equity market implies that $S_t = W_t$. Combining these two conditions, we have $S_t = \frac{1}{a}D_t$, where $a = h_0 + h_1(\psi \log(\delta) - h_t)$. In addition, $S_t = \frac{1}{a}D_t$, implies immediately that $\mu_S = \mu_D + a$ and $\sigma_S = \sigma_D$. Now by applying these results and the equity market equilibrium to equation (37), after some simplifications, we can obtain equations (50) and (51) by collecting terms in v_t and the constant terms. Using equation (39) and equations (50) and (51), it is straightforward to show that the equilibrium risk-free rate and total equity premium are as given in equations (49) and (52). ▮

Similar to Maenhout (2004), the equilibrium equity premium is also given by a CCAPM result [Breeden (1979)] due to the fact that consumption growth and equity returns are by construction perfectly correlated. In the stochastic volatility model, the excess risk premium is given by three components: the market risk premium, the stochastic volatility risk premium, and the model uncertainty risk premium. The second term is the stochastic volatility risk premium and usually takes a positive value due to the negative correlation between volatility and asset returns [Black (1976)] and risk aversion $\gamma > 1$. Equation (49) also shows that the higher the degree of robustness (i.e., the higher the parameter β), the higher the equilibrium equity premium. Therefore, a key empirical prediction of our robust general equilibrium model under stochastic volatility environment is therefore that the price of risk is higher than what would be expected based on both the market risk and the stochastic volatility risk. As in Maenhout (2004), the equilibrium risk-free rate in the stochastic volatility model also depends on time preference, intertemporal substitution and growth, and precautionary savings in response to consumption uncertainty, except that at this time the stochastic volatility risk occurs in the consumption uncertainty. Equation (52) shows that robustness drives down the equilibrium risk-free rate through the precautionary savings channel. Without stochastic volatility, the intuitions for model uncertainty underlying the equilibrium equity premium and the risk-free rate are consistent with the diffusion model with recursive preference in Maenhut (2004).

3. CONCLUSIONS

The discussion about model uncertainty in the expected equity returns and output growth becomes the subject of major disagreement and dispute, because investors' behavior under uncertainty aversion and standard risk aversion exhibits fundamental differences. This paper studies the implications of model uncertainty for the equity premium in a stochastic volatility model by adopting a general equilibrium setting with one representative agent who has a stochastic differential utility. The results show that uncertainty aversion has a contribution to the equilibrium equity premium. In addition, robustness can increase the equilibrium equity premium while lowering the risk-free rate.

REFERENCES

Anderson, E., L. P. Hansen, and T. J. Sargent, 2003. A quartet of semigroups for model specification, robustness, prices of risk, and model detection. *Journal of the European Economic Association* **1**, 68-123.

Black, F., 1976. Studies of stock price volatility changes. *Proceedings of the 1976 meetings of the Business and Economics Statistics Section. American Statistical Association*, pp. 177-181.

Breeden, D., 1979. An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics* **7**, 265-296.

Campbell, J. Y., G. Chacko, J. Rodriguez, and L. M. Viceira, 2004. Strategic asset allocation in a continuous-time VAR model. *Journal of Economic Dynamics and Control* **28**, 2195-2214.

Chen, Z. and L. Epstein, 2002. Ambiguity, risk and asset returns in continuous time. *Econometrica* **70**, 1403-1443.

Duffie, D. and L. Epstein, 1992a. Stochastic differential utility. *Econometrica* **60**, 353-394.

Duffie, D. and L. Epstein, 1992b. Asset pricing with stochastic differential utility. *Review of Financial Studies* **5**, 411-436.

Ellsberg, D., 1961. Risk, ambiguity, and savage axioms. *Quarterly Journal of Economics* **75**, 643-669.

Epstein, L. and S. Zin, 1989. Substitution, risk aversion, and the temporal behavior consumption and asset returns: A theoretical framework. *Econometrica* **57**, 937-969.

Epstein, L. and T. Wang, 1994. Intertemporal asset pricing under Knightian uncertainty. *Econometrica* **62**, 283-322.

Garlappi, L., R. Uppal, and T. Wang, 2007. Portfolio selection with parameter and model uncertainty: A multiple priors approach. *Review of Financial Studies* **20**, 41-81.

Heston, S., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* **6**, 327-344.

Knight, F., 1921. *Risk, Uncertainty and Profit.* Houghton, Mifflin, Boston.

Liu, H., 2010. Robust consumption and portfolio choice for time varying investment opportunities. *Annals of Finance* **6**, 435-454.

Liu, J., J. Pan, and T. Wang, 2005. An equilibrium model of rare-event premia and its implication for option smirks. *Review of Financial Studies* **18**, 131-164.

Lucas, R. E., 1978. Asset prices in an exchange economy. *Econometrica* **46**, 1429-1445.

Maenhout, P. J., 2004. Robust portfolio rules and asset pricing. *Review of Financial Studies* **17**, 951-983.

Merton, R., 1971. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* **3**, 373-413.

Miao J., 2009. Ambiguity, risk and portfolio choice under incomplete information. *Annals of Economics and Finance* **10**, 257-279.

Miao J., 2004. A note on consumption and savings under Knightian uncertainty. *Annals of Economics and Finance* **5**, 299-311.

Pan, J., 2002. The jump-risk premia implicit in option prices: evidence from an integrated time-series study. *Journal of Financial Economics* **63**, 3-50.

Uppal, R. and T. Wang, 2003. Model misspecification and underdiversification. *Journal of Finance* **58**, 2465-2486.

Wang, G. W. and H. F. Zou, 2011. Inflation aversion and the optimal inflation tax. *Annals of Economics and Finance* **12**, 13-20.

Xu, W. D., C. F. Wu, and H. Y. Li, 2010. Robust general equilibrium under stochastic volatility model. *Finance Research Letters* **7**, 224-431.

Zou, H. F., L. T. Gong, and X. S. Zeng, 2011. Inflation aversion. *Annals of Economics and Finance* **12**, 1-11.