ANNALS OF ECONOMICS AND FINANCE **26-2**, 765–823 (2025)

Efficiency in Overlapping Generations Economies with Natural Resources*

Bo Zhang and Zhixiang Zhang†

This paper investigates the Pareto efficiency of competitive equilibria in overlapping-generations (OLG) economies with three productive factors—physical capital, labor, and a natural resource. We derive general criteria for both efficiency and inefficiency by comparing the interest rate with the growth rates of the capital stock, income, and aggregate asset value. Specifically, in the long run, the equilibrium is efficient if any of these growth rates is lower than the interest rate, and inefficient if any exceeds it. These criteria also hold in OLG economies with land. We apply these criteria to several models of resource use, some of which are novel. In one such model, where the resource regeneration function is linear, we establish a threshold for the rate of resource extraction: below this threshold, the equilibrium is efficient; above it, the equilibrium is inefficient. In another novel model featuring a quadratic regeneration function, we introduce a composite capital index. If the labor share is below this index, the equilibrium is efficient; if it exceeds the index, the equilibrium is inefficient.

Key Words: OLG; Dynamic equilibrium; Efficiency; Natural resources.
JEL Classification Numbers: O13, O40, Q20, Q30.

1. INTRODUCTION

In resource economics, the overlapping generations (OLG) model is a fundamental tool to analyze interactions between different generations in resource utilization. It is well known that in OLG economies, even without natural resources, equilibrium can be Pareto inefficient. This means that

* The authors would like to thank Justin Yifu Lin, Min Wang, Xi Weng, Xiaojun Zhao, Pengfei Zhang, Pan Liu, Xinxi Song, Qing Liu, Yulei Luo and Hengfu Zou for their valuable comments and suggestions, and the Key Laboratory of Mathematical Economics and Quantitative Finance (Peking University), Ministry of Education, China, and the National Natural Science Foundation of China (Grant No.72141301) for their financial support.

† Zhang: Corresponding author. School of Economics, Peking University, Beijing 100871, China. E-mail: bozhang@pku.edu.cn; Zhang: China Economics and Management Academy, Central University of Finance and Economics, Beijing, 100081, China. E-mail: zhangzhixiang@cufe.edu.cn.

1529-7373/2025
All rights of reproduction in any form reserved.

the market mechanism does not always result in an efficient allocation of resources. When natural resources are introduced, the problem becomes even more complex.

A fundamental question arises: Given an equilibrium in an OLG economy, is it Pareto efficient? How can we assess the Pareto efficiency of such an equilibrium? Are there appropriate tools or criteria for doing so? This issue is not only of academic importance, but also crucial for policymakers to determine whether market intervention is necessary and, if so, what form that intervention should take. Only when the efficiency issue is identified can we evaluate the role of markets in resource allocation and consider the need for government intervention.

In resource economics, most previous researchers on Pareto efficiency in OLG economies have focused on models with natural resources but without physical capital, primarily analyzing steady-state equilibrium, which is much simpler than dynamic equilibrium. However, when both natural resources and physical capital are considered simultaneously, the problem becomes significantly more complex. In reality, physical capital and natural resources typically coexist in production processes, and their interrelationship has a substantial impact on output. Therefore, it is essential to include both physical capital and natural resources in the analysis. Furthermore, focusing solely on steady-state equilibrium limits the analysis to long-run outcomes, whereas dynamic equilibrium captures the entire process of economic development. Dynamic equilibrium analysis is especially important when a steady state does not exist but a dynamic equilibrium does.

Since Malinvaud (1953), an influential approach to assessing the efficiency of a dynamic economy has been to compare the return on capital with the economy's growth rate—the r-g comparison. This perspective has, in effect, become part of the standard toolkit and is often invoked explicitly or implicitly. Yet most existing results remain narrow in scope, and the literature has not produced clear, general, model-agnostic criteria. This paper develops precise and broadly applicable tests for Pareto efficiency in deterministic OLG economies, unifying and extending the benchmark results along this line.

More concretely, we formulate tests that compare the interest rate with the growth rates of (i) the capital stock, (ii) income, or (iii) aggregate asset value. Intuitively, when the relevant growth rate exceeds the interest rate, the economy exhibits overaccumulation or intertemporal misallocation, implying inefficiency; when it falls short of the interest rate, such overaccumulation is precluded, supporting efficiency (under stated regularity conditions).

The remainder of the paper is organized as follows. First, we conduct a systematic review of the relevant literature. Second, we construct the model framework and present several general criteria. Third, we apply these criteria to three specific examples. Each of these examples represents an interesting model that illustrates some important features of resource utilization. Agnani et al. (2005) introduced a simplified version of the first model, focusing solely on exhaustible resources, whereas the second and third models are original to this paper.

More concretely, in each example, the utility function is log-linear, whereas the resource regeneration function and production function vary. In the first example, the resource regeneration function is linear, and the production function follows a Cobb–Douglas form. In the second example, the resource regeneration function remains linear, but the production function is of the general type of constant elasticity of substitution (CES) beyond Cobb–Douglas. In the third example, the resource regeneration function is logistic, whereas the production function is Cobb-Douglas. Along the way, we also discuss the issues of sustainability and equilibrium stability.

Additionally, through these examples, we address the following question: Does the inclusion of natural resources improve economic efficiency compared to an economy without natural resources? We find that the relationship between natural resources and capital plays a crucial role. In broad terms, if natural resources are substitutable for capital, then the stronger the resource's capacity of regeneration, the more likely their inclusion is to improve the economy's efficiency; if resources are complementary to capital, then a weaker regenerative capacity is more likely to do so. In both cases, the improvement operates by reducing the risk of overaccumulation of capital.

It is important to note that this paper primarily focuses on determining whether a given equilibrium is Pareto efficient, rather than exploring the existence of equilibrium under general conditions. While the existence of equilibrium is certainly a significant issue, typically, existence and efficiency can be analyzed separately. For the three examples discussed, of course, we first need to identify the equilibrium and then evaluate its efficiency.

Developing general criteria to assess equilibrium efficiency in OLG economies is already a challenging task. Due to space limitations, we do not address government intervention in cases of inefficiency. Additionally, we only consider deterministic scenarios and do not tackle the complexities arising from uncertainty.

2. LITERATURE REVIEW

Although this paper studies Pareto efficiency in OLG economies, the modern discussion of dynamic efficiency largely originated in neoclassical Ramsey-type growth models. We therefore begin with that planner literature before turning to competitive OLG equilibria, with and without natural resources.

2.1. Dynamic efficiency in Ramsey-type growth models

Early work on dynamic efficiency analyzed planned economies rather than decentralized equilibria. Malinvaud (1953) introduced the Malinvaud condition, a transversality-style requirement stipulating that the present value of capital must vanish along an efficient program, thereby ruling out "free lunches at infinity." Cass (1972) provided a tractable, price-based criterion for dynamic efficiency: a program is efficient if and only if the series of reciprocals of discount factors diverges. Benveniste and Gale (1975), under some conditions on the production function, extend the Cass criterion: a program is dynamically efficient if the sum of the reciprocals of the norms of present value of capital is divergent. Mitra (1978) transplanted this logic to economies with an exhaustible resource: if the resource is important in production, a competitive program is efficient if and only if a Malinvaud-type terminal condition holds for total assets, i.e., the present value of aggregate wealth tends to zero.

2.2. Pareto efficiency in OLG without natural resources

With finite-lived agents and competitive markets, the model's structure necessitates a focus on intergenerational trade and feasibility. Balasko and Shell (1980) established Cass-type tests for pure-exchange OLG economies: upon suitable assumptions, an equilibrium is Pareto inefficient if and only if a series formed from inverse price norms converges, market language for intertemporal arbitrage at infinity. Wilson (1981) studied environments mixing infinitely and finitely lived agents and derived a sufficient efficiency condition based on the finiteness of the sum of the present value of initial endowments. In stochastic settings, Abel et al. (1989) introduced the net-dividend criterion, offering a practical test that nests earlier intuition. Geanakoplos and Polemarchakis (1991) identified a complementary mechanism: if the first generation owns an asset that produces income in every period, a sufficient condition for efficiency can be obtained. Homburg (1992) provided another factor-income-based sufficiency condition: the present value of wages must vanish along the equilibrium path, an OLG analogue of Malinvaud's insight. Finally, Tirole (1985) showed that with productive and nonproductive (bubbly) assets, OLG economies may feature multiple steady states with different efficiency properties; in such environments, rate comparisons involving the interest rate and demographic growth are informative about bubbles and intergenerational inefficiencies. Collectively, these results translate the conceptual intuition of the transversality condition into market-based tests for OLG equilibria, utilizing prices and income. In addition, they reveal that demographics and asset composition matter in a manner distinct from the outcomes of Ramsey programs.

2.3. Pareto efficiency in OLG with natural resources

Introducing natural resources changes both technology and intertemporal trade in essential ways. Kemp and Long (1979) developed an OLG model with an exhaustible resource but without capital, in which the production function is not homogeneous of degree one, and the resource is not essential for production. The unique equilibrium is inefficient, in which nothing is extracted. Rhee (1991) established a robust sufficiency condition for efficiency when land is important in production (its income share does not vanish) and also showed by counterexample that importance is not necessary. Olson and Knapp (1997), returning to exhaustible resources without physical capital, proved that the resource is ultimately depleted and equilibrium is Pareto efficient even without assuming importance, demonstrating that scarcity alone need not imply inefficiency. For renewable resources, Krautkraemer and Batina (1999) studied the utility of additive logarithmic functions with logistic regeneration and showed that when the share of the output of the resource is small, overaccumulation of the stock can occur, producing inefficient steady states, reversing the Ramsey intuition from “too much capital” to “too much resource”. Koskela et al. (2002), also without capital and under quasi-linear preferences, demonstrated coexistence of a stable and an unstable steady state, with the unstable one always efficient and the stable one potentially inefficient—underscoring that stability and efficiency can diverge. Reintroducing capital, Agnani et al. (2005) examined a Cobb–Douglas/log-utility economy on a balanced growth path and proved social optimality (hence efficiency). In a study of renewable resources (without capital), Farmer and Bednare-Friedl (2017) incorporated logistic growth and harvesting costs that depend on the resource stock. They established that with inverse dependence on stock and certain parameter restrictions, a unique, asymptotically stable steady state can achieve Pareto efficiency despite the resource’s own rate of return being negative. Their work highlights that the structure of costs can supersede simplistic rate-comparison rules. Taken together, these findings demonstrate that the efficiency of OLG equilibria is fundamentally determined by the nature and significance of the resource (e.g., whether it is exhaustible or renewable, and its regeneration technology), as well as by income shares and cost structure. This complexity precludes the use of a single, universally applicable efficiency test.

In summary, across these strands, efficiency consistently appears as “no arbitrage at infinity”, implemented via vanishing present values (Malinvaud; Homburg), divergence tests based on prices (Cass; Balasko–Shell), or dividend-based conditions (Abel et al.). In OLG settings, demographics, asset structure (productive versus bubbly), and the presence and nature of resources complicate how that idea translates into verifiable criteria. The widespread heuristic is to compare the return on capital with the growth rate—the r-g lens—to diagnose overaccumulation and dynamic inefficiency. However, despite its prominence, there is still no clear, general, deterministic criterion that settles efficiency uniformly on the basis of such rate comparisons.

Relatedly, Hellwig (2024) considers stochastic OLG economies that exclude labor and natural resources and examines only the autarkic allocation's interim Pareto efficiency using the r-g comparison; hence, the results do not subsume the deterministic, resource-inclusive economies analyzed here.

In addition, Sachs and Warner (1995, 2025), a seminal contribution to the resource-curse literature, argue, on the basis of a partial cross-country sample, that the impact of resource abundance on economic development is negative. To elucidate the mechanism underlying this impact, they develop, in an appendix, a three-sector OLG model in which a resource boom induces Dutch disease and erodes learning-by-doing in tradables, thereby dampening growth. Subsequent theoretical and empirical work, drawing on different country sets, time windows and measures, and exploring different channels (e.g., Big Push mechanisms[1] and institutional pathways[2]), is mixed, reporting positive, negative, and null effects[3] . Consequently, the "resource curse" remains a great puzzle and an active area for further research.[4] Notably, Sachs and Warner (1995, 2025) do not analyze the efficiency issue, and assessing efficiency is harder in multi-sector OLG economies than in the single-sector settings typically considered in the literature mentioned above[5] .

3. MODEL SETUP

We begin with some preliminary notation. Let $\mathbb{N}$ ($\mathbb{N}_+$) be the set of nonnegative (positive) integers, and $\mathbb{N}_- = \{-1\} \cup \mathbb{N}$. For any natural number n, let $\mathbb{R}^n_{++}$ be the open positive orthant of $\mathbb{R}^n$, that is, $\mathbb{R}^n_{++} = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_i > 0, i = 1, ..., n\}$.

For any two positive dynamic variables x_t and y_t, we use $x_t \sim y_t$ to indicate that y_t/x_t converges to some positive number, as $t \to \infty$.

[1]See, e.g., Sachs and Warner (1999).
[2]See, e.g., Mehlum et al. (2006).
[3]See, e.g., Wick and Bulte (2009) and Ding and Field (2005).
[4] Our paper does not aim to resolve this puzzle.
[5]With Dutch disease and learning-by-doing, the decentralized economy is likely inefficient because of the learning externality, which can be established using the approach developed in this paper.

3.1. The Economy

Consider a two-period OLG economy with natural resources that exist at all points in time within $\mathbb{N}$.

Population: At any time $t \in \mathbb{N}$, a new generation (generation-t) of the population N_t is born, living for two periods. Each individual of generation-t has utility function $U(a_t, b_{t+1})$, where a_t and b_{t+1} represent their consumption at t and $t+1$, respectively, and U is smooth, concave, and strictly increasing with respect to all elements.

Furthermore, at $t = 0$, there is an original generation of size N_{-1}. Each member of this generation lives for just one period with utility function $u(b_0)$, where u is strictly increasing, smooth, and concave, and b_0 is their consumption at $t = 0$.

Endowments: Every young individual is endowed with one unit of labor. Members of the original generation equally share physical capital $\overline{K}_0 > 0$ and natural resource $\overline{S}_0 > 0$ (whether renewable or non-renewable).

Firms: At each time $t \in \mathbb{N}$, there is only one sector comprising numerous homogeneous firms sharing an identical technology represented by production function

$$Y = F^t(K, L, R),$$

where Y is the output of the final good, and K, L and R are the inputs of factors: the physical capital, labor, natural resources, respectively, F^t is the production function at time t, which is homogeneous of degree one, smooth, concave, and strictly increasing with respect to each element. Production functions may change due to technological progress. The final goods can be consumed or invested in physical capital. For simplicity, the depreciation rate for physical capital is assumed to be 1.

Natural resources: Each natural resource, viewed as a unified entity, is extracted and sold. They are not physically divided among the owners. Instead, owners have shared property rights and, therefore, have a equal split in the revenue derived from these resources[6]. Harvesting these resources is cost-free.

With respect to resource transaction and dynamics, we make the assumptions as follows[7]:

At the beginning of each period t, the natural resource (with a stock of S_t) is held by older adults with even property rights. A portion of the resource, R_t, is extracted and sold to firms, and the remaining resources $S_t - R_t$ are sold to young people with even property rights. At the start of the next period, $t + 1$, the resource stock grows to $G(S_t - R_t)$. The function G describes the regeneration of resources. It is defined on $[0, \infty)$,

[6]Tirole(1985) and Rhee (1991), among others, adopt this treatment.
[7]Farmer (2000), among others, uses an alternative approach.

smooth, concave, and nonnegative, with properties $G(0) = 0$, $G'(0) \in (0, \infty]$, $G'(x) > 0$, $\forall x > 0$.

In particular, it refers to the case of exhausted (non-renewable) resources if $G(x) = x$.

The dynamics of the resource are described by

$$S_{t+1} = G(S_t - R_t).$$

Assume that all markets are completely competitive and that every young person has perfect foresight regarding the price system in the next period.

3.2. Efficiency and Social Optimality

The main concern in this paper is Pareto efficiency. Two other related concepts are dynamic efficiency and social optimality. We discuss them separately.

We first give the concepts of allocation and program.

Allocation: A sequence of nonnegative vectors $\{a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$ is called an allocation, if it satisfies the conditions of feasibility: $K_0 = \overline{K}_0$, $S_0 = \overline{S}_0$, and for any $t \in \mathbb{N}$,

$$N_t a_t + N_{t-1} b_t + K_{t+1} \leq F^t(K_t, N_t, R_t),$$

$$S_{t+1} = G(S_t - R_t).$$

Denote the set of all allocations by $\mathscr{A}$. For any allocation $\mathbb{A} = \{a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, and any $t \in \mathbb{N}_-$, denote the utility of generation-t under $\mathbb{A}$ as $U_t(\mathbb{A})$.

Program: A sequence of nonnegative vectors $\{C_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$ is called a program if it satisfies the feasibility conditions: $K_0 = \overline{K}_0$, $S_0 = \overline{S}_0$, and for any $t \in \mathbb{N}$,

$$C_t + K_{t+1} \leq F^t(K_t, N_t, R_t),$$

$$S_{t+1} = G(S_t - R_t).$$

Denote the set of all programs by $\mathscr{P}$.

For any allocation $\{a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, the program $\{C_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, where

$$C_t = N_t a_t + N_{t-1} b_t, \quad \forall t \in \mathbb{N},$$

is called its corresponding program. Here, a_t, b_t, and C_t are the consumption of each young man, the consumption of each old man, and the aggregate consumption at time t, respectively.

Pareto improvement: An allocation $\mathbb{A}$ is Pareto-improved by another allocation $\mathbb{A}'$ if

$$U_t(\mathbb{A}) \leq U_t(\mathbb{A}'), \quad \forall t \in \mathbb{N}_-,$$

with at least one inequality being strict.

Pareto efficiency: An allocation is Pareto efficient, if it cannot be Pareto-improved by any allocation.

The other concept is dynamic efficiency.

Dynamic improvement: A program $\{C_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$ is dynamically improved by another program $\{C'_t, K'_t, S'_t, R'_t\}_{t\in\mathbb{N}}$, if

$$C_t \leq C'_t, \quad \forall t \in \mathbb{N},$$

with at least one inequality being strict.

Dynamic efficiency[8]: A program is dynamically efficient if it cannot be dynamically improved by any program.

An allocation is dynamically efficient if its corresponding program is dynamically efficient.

Clearly, dynamic efficiency is weaker than Pareto efficiency. The converse is not universally true, even if the allocation is an equilibrium allocation (see Section 3.3). Below is a counterexample, the essence of which is the same as in Hilbert's infinite hotel paradox.

There are no natural resources, no population growth, and no technological progress. The production function is $F(K, L) = K + L$, the utility function is $U(a, b) = a + b$, and the initial endowment of capital of the ancestor is $K_0 = 1$. Then the equilibrium allocation $(a_t, b_t, k_t)_{t\in\mathbb{N}}$ (the corresponding price system is $r_t \equiv 0$, $\omega_t \equiv 1$) is dynamically efficient, where

$$a_t = 1, \quad \forall t \in \mathbb{N},$$

$$b_0 = 1, \quad b_t = 0, \quad \forall 1 \leq t \in \mathbb{N},$$

$$k_0 = 1, \quad k_t = 0, \quad \forall 1 \leq t \in \mathbb{N}.$$

But it is not Pareto efficient, because it can be Preto-improved by the following allocation $(a'_t, b'_t, k'_t)_{t\in\mathbb{N}}$:

$$a'_t = 0, \quad \forall t \in \mathbb{N},$$

$$b'_0 = 2, \quad b'_t = 1, \quad \forall 1 \leq t \in \mathbb{N},$$

[8]Some authors call it the dynamic efficiency in the aggregate. See, for example, Miao (2020).

$$k'_0 = 1, \quad k'_t = 0, \quad \forall 1 \leq t \in \mathbb{N},$$

which is itself Pareto efficient[9].

The third concept is social optimality, which is concerned with measuring social welfare. The social welfare is typically measured by a social welfare functional. In most cases, this functional takes the form of a weighted sum of utilities across generations[10]: given a sequence of positive numbers as weights $\lambda = (\lambda_t)_{t \in \mathbb{N}_-}$, for any $\mathbb{A} \in \mathscr{A}$, the social welfare functional is defined as

$$W_\lambda(\mathbb{A}) = \sum_{t=-1}^{\infty} \lambda_t U_t(\mathbb{A}).$$

The most commonly used weights follow an exponential form: for any $t \in \mathbb{N}_-$, $\lambda_t = \varepsilon^t$, where $\varepsilon \in (0,1)$ is known as the social discount factor.

In this paper, we only consider the specific weights of this exponential form, and simply denote it as W_ε.

In particular, if for any $t \in \mathbb{N}$, the utility function for generation-t is of the form $U(a_t, b_{t+1}) = u(a_t) + \rho u(b_{t+1})$, and the utility function for the ancestor is $\rho u(b_0)$, where u is some smooth, concave, and strictly increasing function, then for any $\varepsilon \in (0,1)$, the social welfare functional W_ε can be simplified to a reduced form: for any $\mathbb{A} = (a_t, b_t, K_t, S_t, R_t)_{t \in \mathbb{N}} \in \mathscr{A}$,

$$W_\varepsilon(\mathbb{A}) = \rho u(b_0) + \sum_{t=0}^{\infty} \varepsilon^{t+1} \left(u(a_t) + \rho u(b_{t+1}) \right) = \sum_{t=0}^{\infty} \varepsilon^t (\varepsilon u(a_t) + \rho u(b_t)).$$

Social optimality: An allocation is socially optimal with respect to a social welfare functional W_ε for some $\varepsilon \in (0,1)$ if it maximizes W_ε over $\mathscr{A}$.

Obviously, Pareto efficiency is weaker than social optimality. But the converse is not universally true, even under arbitrary weights.

Social optimality can serve as a useful tool for assessing Pareto efficiency, while also holding intrinsic significance in its own right. From a societal perspective, social optimality offers a framework for determining whether a given allocation is desirable. This criterion operates at a higher level than the Pareto principle, which is often regarded as the most fundamental form of optimality. Consequently, many researchers who address Pareto efficiency also consider social optimality where appropriate. In this paper,

[9]This assertion can be proved by a lemma similar to Lemma 3 in the Appendix. It states that an allocation can be Pareto-improved if and only if the ancestor can be made strictly better off without making anyone else worse off. In allocation $(a'_t, b'_t, k'_t)_{t \in \mathbb{N}}$, the ancestor's consumption is 2, which is already the maximum of the output at time $t = 0$ and of course cannot be improved any more.

[10]In certain scenarios, such a social welfare functional may not be well defined. In these instances, an alternative approach, such as the *overtaking criterion*, may be employed instead.

we apply the concept of social optimality in some specific examples. However, in cases where it is difficult to construct a suitable social welfare functional, we refrain from further analysis.

3.3. Equilibrium

With the final good as the numéraire (with price set to 1), the prices of the physical capital, labor and the natural resource at time $t \in \mathbb{N}$ are denoted as r_t, ω_t, and p_t, respectively, and the consumption of each young individual and the consumption of each old individual at time t are denoted as a_t and b_t, respectively.

Equilibrium: A price system and an allocation, $\{r_t, \omega_t, p_t; a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, with $(1 + r_t, \omega_t, p_t, a_t, b_t, K_t, S_t, R_t) \in \mathbb{R}^8_{++}$ for any $t \in \mathbb{N}$, is called a dynamic equilibrium (or simply an equilibrium), if for any $t \in \mathbb{N}$,

$$\begin{aligned}&(a_t, b_{t+1}, K_{t+1}/N_t, (S_t - R_t))\\ \in\ &\arg \max_{(a,b,s,X)} \{U(a, b) | a + s + p_t X/N_t \leq \omega_t; b = (1 + r_{t+1})s + p_{t+1}G(X)/N_t\}\,;\end{aligned}$$

$$(K_t, N_t, R_t) \in \arg \max_{(K,L,R)} F^t(K, L, R) - (1 + r_t)K - \omega_t L - p_t R;$$

$$K_{t+1} = F^t(K_t, N_t, R_t) - N_t a_t - N_{t-1} b_t, \quad S_{t+1} = G(S_t - R_t).$$

It is easy to verify that along the equilibrium path $\{r_t, \omega_t, p_t; a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$, it holds that for any $t \in \mathbb{N}$,

$$1 + r_{t+1} = \frac{p_{t+1} G'(S_t - R_t)}{p_t}, \tag{1}$$

which is the no-arbitrage condition, implying that the rates of return on investments in any assets (including physical capital and natural resource) are equal.

This no-arbitrage condition can be referred to as the generalized Hotelling rule. It reduces to the classical Hotelling rule (Hotelling(1931)) when $G(x) = x$, corresponding to the case of exhaustible resources.

Existence of equilibrium: Concerning the existence of the equilibrium, it is easy to see that the following basic assertion holds true, which we present as a lemma.

LEMMA 1. *An equilibrium exists if and only if the following system of equations for* $(a_t, b_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$ *with* $K_0 = \overline{K}_0, S_0 = \overline{S}_0$ *has a positive* *solution:*

$$
\begin{aligned}
&F_K^t(K_{t+1}, N_{t+1}, R_{t+1}) = \frac{F_R^{t+1}(K_{t+1}, N_{t+1}, R_{t+1})}{F_R^t(K_t, N_t, R_t)} G'(S_t - R_t),\\
&F_K^t(K_{t+1}, N_{t+1}, R_{t+1}) = \frac{U_a(a_t, b_{t+1})}{U_b(a_t, b_{t+1})},\\
&F^t(K_t, N_t, R_t) = N_t a_t + N_{t-1} b_t + K_{t+1},\\
&N_{t-1} b_t = K_t F_K^t(K_t, N_t, R_t) + S_t F_R^t(K_t, N_t, R_t),\\
&S_{t+1} = G(S_t - R_t).
\end{aligned}
$$

The first equation is the previously mentioned generalized Hotelling rule (1); the second reflects the equality of MRS (marginal rate of substitution) and MRT (marginal rate of transformation) in each period; the third is the feasibility condition; the fourth indicates that older people consume all of their assets; and the fifth describes the dynamic equation for the resource stock.

Because $(N_t)_{t\in\mathbb{N}}$ is given exogenously, and $(a_t, b_t)_{t\in\mathbb{N}}$ can be derived from $(K_t, S_t, R_t)_{t\in\mathbb{N}}$, then the above system of equations for $(a_t, b_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$ can be equivalently transformed to a system of equations for $(K_t, S_t, R_t)_{t\in\mathbb{N}}$. According to the implicit function theorem, under certain conditions (e.g., the corresponding Jacobian matrix is nondegenerate), explicit recursive equations can be derived:

$$
\begin{aligned}
&K_{t+1} = \varphi(K_t, S_t, R_t, N_t, N_{t+1}),\\
&S_{t+1} = G(S_t - R_t),\\
&R_{t+1} = \psi(K_t, S_t, R_t, N_t, N_{t+1}),
\end{aligned}
$$

where φ, ψ are some functions. In this case, an equilibrium exists, if and only if there exists an $R_0 > 0$, ensuring the entire trajectory of $(K_t, S_t, R_t)_{t\in\mathbb{N}}$ remains in $\mathbb{R}^3_{++}$[11].

For simplicity, we say the equilibrium is dynamically efficient (Pareto efficient), if the corresponding equilibrium allocation is dynamically efficient (Pareto efficient).

4. GENERAL RESULTS

Given an equilibrium of the above economy $\{r_t, \omega_t, p_t; a_t, b_t; K_t, S_t, R_t\}_{t\in\mathbb{N}}$, the question we are concerned with is: is it efficient? In order to answer

[11] At times, this dynamical system can be further transformed to a dynamical system of $(k_t, s_t, z_t)_{t\in\mathbb{N}}$, where k_t, s_t, z_t are the capital, resource stock, and resource extraction per effective labor.

this question, we first need to introduce some basic notations. Afterward, we will provide some criteria to assess efficiency.

For any $t \in \mathbb{N}$, denote the total output as

$$Y_t = F^t(K_t, N_t, R_t).$$

For any $t \in \mathbb{N}_+$, define the market discount factor from time t to time 0 as

$$D_t = \prod_{s=1}^{t}(1 + r_s)^{-1},$$

and set $D_0 := 1$. For convenience, let $D_{-1} = 1 + r_0$.

We know that by the generalized Hoteling rule, for any $t \in \mathbb{N}$,

$$D_{t+1}p_{t+1}G'(S_t - R_t) = D_t p_t.$$

For any $t \in \mathbb{N}$, denote the total value of assets held by all the old people at time t as

$$V_t = (1 + r_t)K_t + p_t S_t,$$

the combined total investments made by all the young people of generation-t as

$$M_t = K_{t+1} + p_t(S_t - R_t),$$

the total income from both labor and investment in natural resources of generation-t as

$$I_t = \omega_t N_t + \frac{p_{t+1}}{1 + r_{t+1}} S_{t+1} - p_t(S_t - R_t).$$

At time t, the old people hold the assets (K_t, S_t). Through market transactions, they can obtain the total revenue V_t and consume it. Clearly, $V_t = N_{t-1}b_t$.

For any $t \in \mathbb{N}$, when considering society as a whole, define the dividend as

$$Z_t = V_t - M_t = (1 + r_t)K_t - K_{t+1} + p_t R_t.$$

For any $t \in \mathbb{N}$, denote the growth rate of total income and the growth rate of the physical capital stock (the growth rate of capital, for short) at time t by

$$i_t = \frac{I_t}{I_{t-1}} - 1, \quad j_t = \frac{K_{t+1}}{K_t} - 1,$$

respectively.

Of course, if there is no natural resource, the above concepts reduce respectively to

$$V_t = (1 + r_t)K_t, \quad M_t = K_{t+1}, \quad I_t = \omega_t N_t, \quad Z_t = (1 + r_t)K_t - K_{t+1}.$$

4.1. Main Criteria

We discuss the dynamic efficiency and Pareto efficiency separately.

4.1.1. Dynamic Efficiency

We provide a criterion for dynamic efficiency.

THEOREM 1. *The equilibrium is dynamically efficient, if*

$$\underline{\lim}_{t\to\infty} D_t V_t = 0. \tag{2}$$

Remark 1. Condition (2) is the Malinvaud-type condition. Malinvaud (1953) originally formulates it in a neoclassical growth model without natural resources[12]. Mitra (1978) extends Malinvaud's result to cases with exhaustible resources. We further extend it to cases with any type of natural resources. This condition means that the total wealth is eventually exhausted.

In order to get the converse of the above theorem, we make the following assumptions:

A1. The regeneration function for the natural resource is linear.

A2. The natural resource is important in production relative to labor, meaning that[13]

$$\underline{\lim}_{t\to\infty} \frac{p_t R_t}{\omega_t N_t} > 0.$$

THEOREM 2. *Under assumptions* **A1,A2**, *if the equilibrium is dynamically efficient, then*

$$\lim_{t\to\infty} D_t V_t = 0.$$

[12] See also Theorem 1 in Becker and Mitra (2012).

[13] A stronger version of this condition is

$$\inf_{t\in\mathbb{N},(K_t,N_t,R_t)\in\mathbb{R}^3_{++}} \frac{R_t F_R^t(K_t, N_t, R_t)}{F^t(K_t, N_t, R_t)} > 0.$$

That is, the minimum income share of natural resource at all times is away from zero.

Remark 2. Mitra (1978) obtains this result for the case of exhaustible resources. We extend it to cases involving natural resources with arbitrary linear regeneration functions.

4.1.2. Pareto Efficiency

We provide a criterion for Pareto efficiency.

THEOREM 3. *The equilibrium is Pareto efficient if*

$$\varliminf_{t\to\infty} D_t\omega_t N_t = 0. \tag{3}$$

Remark 3. Homburg (1992) first introduced condition (3) without considering natural resources[14]. We extend it to cases involving natural resources. The condition means that all income derived from labor is eventually exhausted.

To some extent, we can say that the criterion in Theorem 3 involves comparing the growth rate of income from labor (e.g., wages) with the interest rate. Clearly, condition (3) is weaker than

$$\varlimsup_{t\to\infty} \frac{1+i'_t}{1+r_t} < 1,$$

where $i'_t = (\omega_t N_t)/(\omega_{t-1}N_{t-1}) - 1$ is the growth rate of total wages.

And obviously, condition (3) is also weaker than the Wilson-type condition:

$$\sum_{t=0}^{\infty} D_t Y_t < \infty.$$

We now provide a criterion for Pareto inefficiency. We need the following assumption.

A3. The technological progress is Harrod-neutral, that is, for any $t \in \mathbb{N}$, $F^t(K,L,R) = F(K,B_tL,R)$ for some function F, which is homogeneous of degree one, smooth, concave, and strictly increasing with respect to each element; and $B_t = (1+\nu)^t$, $N_t = (1+n)^t$, where $\nu \geq 0$, $n > -1$ are constants; $\sup_t z_t < \infty$; $\lim\limits_{k\to\infty} f_k(k,z) < \mu =: (1+\nu)(1+n)$ uniformly for z in any bounded interval; and either $\liminf\limits_{t\to\infty} k_t > 0$, or $f_k(0,z) < \infty$ for any $z \geq 0$, where $f(k,z) = F(k,1,z)$, $k_t = K_t/(B_tN_t)$, $z_t = R_t/(B_tN_t)$[15].

---

[14]See Theorem 1 in Homburg (1992), which excludes natural resources, although he later addresses land in subsequent chapters.

[15]The condition $\sup_t z_t < \infty$ means that the resource extraction per effective labor is bounded above. A stronger version is $\sup_t S_t/(B_tN_t) < \infty$, which is determined

THEOREM 4. *Under assumption* **A3**, *the equilibrium is Pareto inefficient if*

$$\varliminf_{t\to\infty} \frac{1+j_t}{1+r_t} > 1. \tag{4}$$

Remark 4. This result implies that if the growth rate of the capital stock exceeds the interest rate in the long run, the economy experiences overaccumulation of capital, leading to inefficiency. Roughly, Abel et al. (1989) initially presented this criterion, without considering natural resources. We extend it to cases where natural resources are taken into account under some conditions on production functions.

Remark 5. (Further discussion on Abel et al. (1989)) Abel et al. (1989) introduces the net dividend criterion for Pareto inefficiency, stating that an equilibrium allocation is Pareto inefficient if

$$\frac{Z_t}{M_t} \leq -\epsilon, \quad \forall t \in \mathbb{N}, \tag{5}$$

for some $\epsilon > 0$. This is equivalent to

$$\frac{1+j_t}{1+r_t} \geq 1+\epsilon', \quad \forall t \in \mathbb{N},$$

for some $\epsilon' > 0$.

Clearly, our condition (4) is weaker than (5) of Abel et al. (1989). Instead of requiring the capital growth rate to consistently exceed the interest rate throughout the entire process of economic development, our condition only demands this in the long run, as time approaches infinity. What is the intuition behind our result? From a certain point in time, say T, onward, if the capital growth rate exceeds the interest rate, this is sufficient to ensure inefficiency. There is no need to impose this condition from the very beginning (i.e., $T = 0$). The portion of the equilibrium allocation before T can remain unchanged, while the allocation after T can be Pareto improved by reducing capital and increasing consumption. This improvement is possible as long as the capital growth rate continues to exceed the interest rate from time T onward.

exogenously by the regeneration function, technological progress, and the population growth.

Abel et al. (1989) also presents the net dividend criterion for Pareto efficiency, stating that an equilibrium allocation is Pareto efficient if

$$\frac{Z_t}{M_t} \geq \epsilon, \quad \forall t \in \mathbb{N}, \tag{6}$$

for some $\epsilon > 0$. This is equivalent to

$$\frac{1 + j_t}{1 + r_t} \leq 1 - \epsilon', \quad \forall t \in \mathbb{N},$$

for some $\epsilon' > 0$. However, Chattopadhyay (2008) disproves this criterion by providing a counterexample[16].

At the end of this subsection, we attempt to investigate the converse of Theorem 3 to some extent.

Theorem 3 gives a criterion of Pareto efficiency in terms of income from labor (i.e., the wages), but, it is not necessary[17].

Now, we try to provide a criterion for Pareto inefficiency in terms of income. However, we find that we need to modify the income from labor to the total income, and compare the growth rate of the total income with the interest rate, which is similar to the criterion in Theorem 4 comparing the growth rate of capital and the interest rate.

Additionally, to derive a concise form of such a criterion, here we only consider the case of log-linear utility function[18]. But the regeneration and production functions remain arbitrary, ensuring that the result is still relatively general.

THEOREM 5. *Suppose $U(a, b) = \ln a + \rho \ln b$, where $\rho \in (0, 1)$ is a constant. Then, the equilibrium is Pareto inefficient if*

$$\varliminf_{t \to \infty} \frac{1 + i_t}{1 + r_t} > 1. \tag{7}$$

Remark 6. Condition (7) implies that if total income grows too rapidly, eventually surpassing the interest rate in the long run, valuable resources will remain underutilized, leading to inefficiency. This introduces a new criterion for assessing Pareto inefficiency. The underlying mechanism in

[16]After adding an additional condition, condition (6) is really sufficient for the Pareto efficiency of the equilibrium. See Corollary 3 below.

[17]Rhee (1991) presents a counterexample for an OLG economy with land.

[18]Other forms of utility functions can be considered, but the resulting formulas would be complex and less elegant, so they are not presented here.

this new criterion is the same as in Hilbert's infinite hotel paradox. It will be applied to prove the Pareto inefficiency of equilibrium in an example involving a quadratic regeneration function of the resource, where assumption A3 is not satisfied, making the criterion in Theorem 4 insufficient to guarantee the Pareto inefficiency of the equilibrium.

4.2. Corollaries

From the above theorems, we immediately have the following corollaries.

Corollary 1. *Under the assumptions* **A1,A2***, the equilibrium is Pareto efficient if and only if* $\lim_{t\to\infty} D_t V_t = 0$.

Remark 7. Under **A1,A2**, the condition $\lim_{t\to\infty} D_t V_t = 0$ is a complete characterization for Pareto efficiency of the equilibrium.

Corollary 2. *If*

$$\overline{\lim_{t\to\infty}} \frac{R_{t+1}}{R_t G'(S_t - R_t)} < 1, \tag{8}$$

$$\overline{\lim_{t\to\infty}} \frac{p_t R_t}{\omega_t N_t} > 0, \tag{9}$$

then, the equilibrium is Pareto efficient.

Remark 8. Condition (8) indicates that for large t,

$$\frac{R_{t+1}}{R_t} < G'(S_t - R_t),$$

which means that the growth speed of harvesting is lower than the marginal regeneration capacity. In other words, the natural resource is not extracted too quickly; condition (9) means that in the long run, in production, the resource share is not nil, compared with the labor share. Condition (9) is weaker than the assumption **A2**.

Corollary 3. *If*

$$\overline{\lim_{t\to\infty}} \frac{1+j_t}{1+r_t} < 1, \tag{10}$$

$$\overline{\lim_{t\to\infty}} \frac{(1+r_t)K_t}{\omega_t N_t} > 0, \tag{11}$$

the equilibrium is Pareto efficient.

Remark 9. Roughly speaking, condition (10) indicates that the growth rate of capital is less than the interest rate, which can be seen as the reverse inequality of condition (4); condition (11) means that the physical capital is important in production relative to labor. This result can be seen as the converse of Theorem 4 to some extent and also as a modified version of the dividend criterion for Pareto efficiency of Abel et al. (1989)[19].

By combining Theorem 4 and Corollary 3, we can infer that the long-term relationship between the growth rate of capital and the interest rate is crucial in determining economic efficiency. Roughly, if the growth rate of capital consistently exceeds the interest rate in the long run, it may result in over-accumulation of capital, leading to inefficiency. Conversely, if the growth rate of capital remains below the interest rate in the long run, capital accumulation is more controlled, promoting efficiency. This conclusion aligns with our intuitive understanding. In summary, comparing the long-term growth rate of capital to the interest rate provides a meaningful approach to assessing economic efficiency. Of course, for rigorousness, certain preassumptions would need to be made.

Corollary 4. *Under assumption* **A3***, furthermore, suppose that as* $t \to \infty$*,*

$$\frac{K_t}{B_t N_t} \to k^* > 0, \quad r_t \to r^* > -1, \quad \frac{\omega_t}{B_t} \to \omega^* > 0,$$

where k^*, r^*, ω^* *are constants. Then, the equilibrium is Pareto efficient if* $r^* > n'$*; it is Pareto inefficient if* $r^* < n'$*, where* $n' = (1+\nu)(1+n) - 1$ *is the growth rate of capital in the long run.*

Remark 10. Theorems 1, 3, 4, and 5, and Corollaries 3 and 4 remain valid in the absence of natural resources. This is because the proofs of these results do not rely on the assumption that "the production function at any time is strictly increasing with respect to the natural resource." (See Section 3.1.). However, Theorem 2 does require this assumption for its validity.

[19]See Remark 5.

5. APPLICATIONS

In this section, we apply the above general criteria to several examples where the regeneration function is either linear or quadratic.

The quadratic regeneration function is the simplest example to illustrate the characteristic that any renewable resource has a finite environmental carrying capacity. Consequently, it is widely used in resource economics.

In contrast, the linear regeneration function is typically used for non-renewable resources, resources that degenerate exponentially, or idealized renewable resources with infinite environmental carrying capacity.

In all of these concrete examples, we first need to identify the equilibrium (or equilibria, if multiple exist) and then evaluate its efficiency. This is where our criteria come into play. While finding the equilibrium can be challenging, our primary focus is on assessing its efficiency.

Moreover, none of these examples is trivial; each represents an intriguing model in resource economics, highlighting some important features of resource use. Specifically, there are three models presented here. A simplified version of the first model appeared in Agnani et al. (2005); the other two are original to this paper.

In all examples in this section, we assume that the utility function is as follows: $U(a,b) = \ln a + \rho \ln b$, where $\rho \in (0,1)$ is a constant.

5.1. Linear Regeneration Function, Cobb–Douglas Production Function

Assume

$$G(x) = \eta x, \quad F^t(K,L,R) = A_t K^\alpha L^\beta R^\gamma, \quad \forall t \in \mathbb{N},$$

where $A_t > 0$, $\eta > 0$, $0 < \alpha, \beta, \gamma < 1$ are constants, satisfying $\alpha+\beta+\gamma = 1$. The parameter η can be called the intensity of the regeneration of the resource.

Agnani et al. (2005) investigate the case of an exhaustible resource where $\eta = 1$. However, they implicitly assume the existence of equilibrium and further assume that the economy follows an exact balanced growth path, with both A_t and N_t growing at exogenously given rates. Under such narrowly defined conditions, they discuss issues of Pareto efficiency, sustainability, and social optimality.

In contrast, we rigorously prove the existence and uniqueness of equilibrium, as well as its Pareto efficiency and social optimality, while also examining the sustainability issue in a more general context.

Mourmouras (1991) demonstrates the existence of equilibrium and addresses sustainability in the absence of technical progress and population change.

Our results in this subsection fully encompass both Agnani et al. (2005) and Mourmouras (1991).

For use in the sequel, let $\theta = \rho\beta/(1+\rho)$, $\delta = 1-\tau$, and $\tau \in (0,1)$ be determined uniquely by

$$\tau = \frac{\gamma + \alpha\tau}{\theta + \gamma + \alpha\tau}.$$

5.1.1. Equilibrium Existence, Uniqueness and Efficiency

By the definition of equilibrium, one can easily verify that the equilibrium exists and is unique if and only if the following system of equations for $(K_t, S_t, R_t)_{t\in\mathbb{N}}$ with $K_0 = \overline{K}_0$ and $S_0 = \overline{S}_0$ has a unique positive solution:

$$K_{t+1} = \frac{Y_t}{R_t}\left[(\theta+\gamma)R_t - \gamma S_t\right], \tag{12}$$

$$S_{t+1} = \eta(S_t - R_t), \tag{13}$$

$$R_{t+1} = \frac{\eta}{\alpha}\left[(\theta+\gamma)R_t - \gamma S_t\right], \tag{14}$$

where $Y_t = A_t K_t^{\alpha} N_t^{\beta} R_t^{\gamma}$. And, obviously, this system of equations has a unique positive solution if and only if the planar dynamical system for $(S_t, R_t)_{t\in\mathbb{N}}$ with $S_0 = \overline{S}_0$, described by (13) and (14), has a unique positive solution.

Following this line of reasoning, we can derive

Proposition 1. *The equilibrium exists and is unique, in which for any $t \in \mathbb{N}$, $N_t a_t = \frac{\beta}{1+\rho} Y_t$, $N_{t-1} b_t = (\alpha + \gamma/\tau) Y_t$, $R_t = \tau S_t$, $K_{t+1} = \alpha\delta Y_t$, where $Y_t = A_t K_t^{\alpha} N_t^{\beta} R_t^{\gamma}$.*

Remark 11. The resource extraction rate is constant τ, and the total consumption of the young, the total consumption of the old, and the investment in capital are all proportional to the total output, each by a constant ratio.

Proposition 2. *The equilibrium is Pareto efficient.*

Remark 12. (Comparison between this model and the classical Diamond OLG model) The case $\gamma = 0$ corresponds to the classical Diamond

OLG model, in which for any $t \in \mathbb{N}$,

$$K_{t+1} = \theta Y_t.$$

Thus,

$$\frac{D_{t+1}Y_{t+1}}{D_tY_t} = \frac{Y_{t+1}}{(1+r_{t+1})Y_t} = \frac{K_{t+1}}{\alpha Y_t} = \frac{\theta}{\alpha},$$

$$\frac{1+j_t}{1+r_t} = \frac{K_{t+1}}{(1+r_t)K_t} = \frac{K_{t+1}}{\alpha Y_t} = \frac{\theta}{\alpha},$$

$$\frac{1+i_{t+1}}{1+r_{t+1}} = \frac{Y_{t+1}}{(1+r_{t+1})Y_t} = \frac{K_{t+1}}{\alpha Y_t} = \frac{\theta}{\alpha},$$

Therefore, if $\alpha > \theta$, then

$$\lim_{t\to\infty} D_t\omega_t N_t = \beta \lim_{t\to\infty} D_tY_t = \beta Y_0 \lim_{t\to\infty} \left(\frac{\theta}{\alpha}\right)^t = 0.$$

Therefore, by Theorem 1, the equilibrium is Pareto efficient.

If $\alpha < \theta$,

$$\lim_{t\to\infty} \frac{1+j_t}{1+r_t} = \lim_{t\to\infty} \frac{1+i_t}{1+r_t} = \frac{\theta}{\alpha} > 1.$$

Therefore, by Theorem 2 or 3, the equilibrium is Pareto inefficient.

In contrast, in the cases of $\gamma > 0$,

$$\frac{1+j_t}{1+r_t} = \frac{K_{t+1}}{(1+r_t)K_t} = \frac{K_{t+1}}{\alpha Y_t} = \delta < 1.$$

This implies that the growth rate of capital is always lower than the interest rate, preventing overaccumulation of capital.

5.1.2. Social Optimality

We will use the following assumption.

A4. $A_t = (1+g)^t, \quad N_t = (1+n)^t, \quad \forall t \in \mathbb{N}$, where $g \geq 0, n > -1$ are constants.

PROPOSITION 3. *Under assumption* **A4**, *the equilibrium allocation is socially optimal with respect to* W_δ.

In the previous subsection, we demonstrated Pareto efficiency without assumption **A4**. With this assumption, however, we can derive a stronger result. Here, δ represents the social discount factor embedded in the market system, whereas ρ corresponds to the individual discount factor.

The social discount factor is determined by the aggregation of individual discount rates across all economic agents in the economy, reflecting the underlying economic structure, including aspects such as resource availability and the relative significance of labor.

Furthermore, δ increases with respect to ρ, indicating a positive correlation between the social and individual discount factors.

Which is larger, δ or ρ? The answer primarily depends on the labor share, β. If β is sufficiently small, then $\delta < \rho$; if β is sufficiently large, then $\delta > \rho$. Specifically, when $\beta = 0$, we have $\delta = 0$, and when $\beta = 1$, we have $\delta = 1$. This suggests that the greater the role of labor in production, the less the "social planner" discounts the future.

Additionally, because $\delta = 1 - \tau$, where τ represents the optimal resource extraction rate in the market, a heavier discounting of the future by the "social planner" implies more rapid resource extraction.

Remark 13. It is easy to verify that under assumption **A4**, the equilibrium path converges to a balanced growth path (BGP), and it is exactly a BGP if and only if

$$\phi K_0^{\beta+\gamma} = \alpha\delta\left(\tau S_0\right)^{\gamma},$$

where $\phi := \big((1+\lambda)(1+n)^{\beta}(\eta\delta)^{\gamma}\big)^{1/(\beta+\gamma)} - 1$.

5.1.3. Sustainability

Denote the output per capita at time t as $y_t = Y_t/N_t$. Because

$$Y_{t+1} = A_{t+1}(\alpha\delta Y_t)^{\alpha} N_{t+1}^{\beta} R_{t+1}^{\gamma},$$

then, $y_{t+1} = m_t y_t^{\alpha}$, where

$$m_t \sim \frac{A_{t+1} N_t^{\alpha}}{N_{t+1}^{\alpha+\gamma}} (\eta\delta)^{\gamma t}.$$

Clearly, the behavior of the economy depends on the long-term behavior of m_t. If $\lim_{t\to\infty} m_t = \infty$, the economy grows without bound. If $\lim_{t\to\infty} m_t = m$ for some $m > 0$, the economy converges to a finite level. If $\lim_{t\to\infty} m_t = 0$, the economy contracts, leading to a collapse. If $(m_t)_{t\in\mathbb{N}}$ does not converge, the economy exhibits fluctuations.

In particular, under **A4**, we have

$$m_t \sim h^t, \quad h := (1+g)\left(\frac{\eta\delta}{1+n}\right)^{\gamma}.$$

Here, h is a composite index, indicating the extent of sustainability. If $h > 1$, the economy expands without bound. If $h = 1$, the economy converges to a finite level. If $h < 1$, the economy contracts.

For example, for a given technical growth rate $g > 0$, if the resource share γ is sufficiently small, then the economy will expand without bound.

For given g, n, η, γ, if the discount factor ρ or the labor share β is sufficiently small, then the resource harvesting rate τ is sufficiently close to 1, and δ is sufficiently low, leading to $h < 1$, and consequently, the economy will contract.

5.2. Linear Regeneration Function, CES Production Function

Now, consider other types of CES functions beyond Cobb-Douglas. More precisely, assume $G(x) = \eta x$, and for any $t \in \mathbb{N}$,

$$F^t(K, L, R) = (\alpha K^\sigma + \beta L^\sigma + \gamma R^\sigma)^{1/\sigma}, \quad N_t = (1+n)^t,$$

where[20] $0 \neq \sigma < 1, \eta > 0, n \geq 0,\ 0 < \alpha, \beta, \gamma < 1$ are given constants, satisfying $\alpha + \beta + \gamma = 1$.

To the best of our knowledge, this model in resource economics is novel and has not been covered in existing literature.

Denote the capital, resource stock, and resource extraction per capita respectively as

$$k_t = \frac{K_t}{N_t}, \quad s_t = \frac{S_t}{N_t}, \quad z_t = \frac{R_t}{N_t}.$$

It is easy to see that the equilibrium exists if and only if the following three-dimensional difference dynamical system $\mathscr{D}$ of $(k_t, s_t, z_t)_{t\in\mathbb{N}}$ with given $k_0 > 0, s_0 > 0$ has positive solution:

$$k_{t+1} = \frac{1}{1+n}\left(\alpha k_t^\sigma + \beta + \gamma z_t^\sigma\right)^{(1-\sigma)/\sigma}\left[\theta - \frac{\gamma(s_t - z_t)}{z_t^{1-\sigma}}\right], \tag{15}$$

$$s_{t+1} = \frac{\eta}{1+n}(s_t - z_t), \tag{16}$$

$$z_{t+1} = \frac{1}{1+n}\left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)}\frac{z_t}{\alpha k_t^\sigma + \beta + \gamma z_t^\sigma}\left[\theta - \frac{\gamma(s_t - z_t)}{z_t^{1-\sigma}}\right], \tag{17}$$

where $\theta = \rho\beta/(1+\rho)$.

[20] When $\sigma = 0$, it will reduce to the Cobb–Douglas case, which is fully analyzed in Section 5.1. In this subsection, we focus exclusively on the case where $\sigma \neq 0$, except in Section 5.2.3, where we discuss the role of σ in shaping the behavior of the economy.

Clearly, any positive solution of $\mathscr{D}$ must meet the feasibility condition that for any $t \in \mathbb{N}$,

$$(s_t, z_t) \in \Theta =: \left\{ (s, z) \middle| 0 < z < s < z + \frac{\theta}{\gamma} z^{1-\sigma} \right\}.$$

All other paths will cross out of the region Θ in a finite time, leading to the collapse of the system.

A steady state of $\mathscr{D}$ is called nontrivial if it lies in $\mathbb{R}^3_{++}$. For convenience, let

$$\nu := \frac{\rho}{1+\rho} \left[\left(\frac{\eta^\sigma}{\alpha} \right)^{1/(1-\sigma)} - 1 \right].$$

Clearly, $\nu > (=, <) 1$ if and only if $\alpha < (=, >) \eta^\sigma \left(2 + \rho^{-1}\right)^{\sigma - 1}$.

Lemma 2. *For the dynamical system $\mathscr{D}$, there exists a unique nontrivial steady state if and only if*

$$\frac{1+n}{\eta} < \min\{1, \nu\}. \tag{18}$$

We observe that when $\sigma > 0$, condition (18) implies that η is large, whereas when $\sigma > 0$, condition (18) means that η lies within a certain interval, neither too large nor too small.

In the sequel, we only consider the possible equilibrium paths satisfying the limit condition: there exist $\hat{k}, \hat{s}, \hat{z} \in [0, \infty]$ and $\epsilon \in [0, \theta]$ such that

$$\lim_{t \to \infty} (k_t, s_t, z_t) = (\hat{k}, \hat{s}, \hat{z}),$$

$$\lim_{t \to \infty} \frac{s_t - z_t}{z_t^{1-\sigma}} = \frac{\theta - \epsilon}{\gamma}. \tag{19}$$

This ϵ can be interpreted as a measure of the speed of resource harvesting, referred to as the harvesting speed indicator, for short.

There are only three possible cases: (i) $\hat{s} = \hat{z} = 0$; (ii) $\hat{s}, \hat{z} \in (0, \infty)$; (iii) $\hat{s} = \hat{z} = \infty$, and correspondingly, the equilibrium is called equilibrium of type I, type II, and type III, respectively. And, obviously, if $\hat{s}, \hat{z} \in (0, \infty)$, then, $\hat{k} \in (0, \infty)$.

Concerning the equilibrium of type II, from Lemma 2, one can easily obtain

Proposition 4. *The Type II equilibrium exists uniquely if and only if condition (18) is satisfied.*

Concerning the efficiency, we have

Proposition 5. *The type II equilibrium is Pareto efficient.*

Therefore, the steady state equilibrium is always Pareto efficient if it exists.

In the sequel, we will examine the existence and efficiency of type I and type III equilibria in two distinct cases: $\sigma \in (0,1)$ and $\sigma < 0$. For the sake of completeness, when presenting results regarding the existence of equilibrium, we will also include type II equilibria.

5.2.1. $\sigma \in (0,1)$ sigma in (0,1)

For any $k \geq 0$, define

$$\pi(k) = (1+n)k\left(\alpha k^{\sigma} + \beta\right)^{(\sigma-1)/\sigma}.$$

It is easy to verify that π is strictly increasing.

The meaning of π is as follows. For a type I equilibrium, there is a $\epsilon \in [0,\theta]$ such that condition (19) holds, and then, by letting $t \to \infty$ in (15), we obtain that the limit capital per capita k satisfies

$$k = \frac{\epsilon}{1+n}(\alpha k^{\sigma} + \beta)^{(1-\sigma)/\sigma},$$

which implies $\epsilon = \pi(k)$. Therefore, π represents the one-to-one relationship between the harvesting speed indicator and the limit capital per capita, and this relationship is positive. In other words, the faster the resource extraction, the higher the limit capital per capita.

Concerning the existence of equilibrium, we have

Proposition 6. *(i) No type III equilibrium exists.*

(ii) If $\frac{1+n}{\eta} < \min\{1,\nu\}$, then there exists unique equilibrium, which is of type II.

(iii) If $\nu \leq \frac{1+n}{\eta} < 1$, then there exists a unique type I equilibrium with limit capital per capita being $\pi^{-1}(\theta)$.

(iv) If $\frac{1+n}{\eta} \geq 1$, then there exists a continuum of type I equilibria. More precisely, there exists an interval $[\underline{z},\overline{z}]$ such that any $z_0 \in [\underline{z},\overline{z}]$ induces a type I equilibrium. Accordingly, there exists a $\overline{k} \in (0,\pi^{-1}(\theta)]$ such that for

any $k \in [0, \overline{k}]$*, there is a type I equilibrium of with the limit capital per capita being* k*. The more* z_0*, the more the limit capital per capita* k*.*

Remark 14. If the regeneration capacity is relatively large compared with the population growth rate and there is no steady state, then there is a unique equilibrium in which the resource is harvested as quickly as possible (the harvesting speed indicator is θ) and finally the resource stock tends to zero, and correspondingly, the capital per capita tends to the largest possible value $\pi^{-1}(\theta)$. If the regeneration capacity is relatively small compared with the population growth rate, then there is a continuum of equilibria.

Concerning the efficiency of equilibrium, we have the following results.

PROPOSITION 7. *(i) If* $\eta > 1 + n$*, then the unique equilibrium is Pareto efficient.*

(ii) If $\eta \le 1 + n$*, then there exists a* $z_* \in [\underline{z}, \overline{z}]$ *such that for any* $z_0 \in [\underline{z}, z_*)$*, the corresponding equilibrium of type I is Pareto efficient; for any* $z_0 \in (z_*, \overline{z}]$*, the corresponding equilibrium of type I is Pareto inefficient. Accordingly, there exists a* $\underline{k} \in (0, \overline{k}]$ *such that for any* $k \in [0, \underline{k})$*, the corresponding equilibrium of type I is Pareto efficient; for any* $k \in (\underline{k}, \overline{k}]$*, the corresponding equilibrium of type I is Pareto inefficient.*

Remark 15. This implies that if the regeneration capacity is relatively large compared with the population growth rate, then, the unique equilibrium is Pareto efficient.

If the regeneration capacity is relatively small compared with the population growth rate, the resource stock per capita will tend to zero, and there exists a threshold for the initial resource extraction z_0 (accordingly, there exist a threshold for the harvesting speed indicator ϵ and a threshold for limit capital per capita k), below which the economy is Pareto efficient and above which it becomes inefficient. In other words, the slower the resource extraction, the higher the likelihood that the economy is Pareto efficient.

5.2.2. $\sigma < 0$

Denote

$$\begin{aligned} \epsilon_* &:= (1+n)\beta(1-\sigma^{-1})\left[\alpha(1-\sigma)\right]^{-1/\sigma}, \\ \alpha^* &:= (1-\sigma)^{-1}\left[(1+n)(1-\sigma^{-1})\frac{1+\rho}{\rho}\right]^{\sigma}. \end{aligned}$$

Clearly, $\theta > (=, <)\epsilon_*$ if and only if $\alpha < (=, >)\alpha^*$.

Concerning the existence of equilibrium, we have the following proposition.

PROPOSITION 8. *(i) If $\frac{1+n}{\eta} \geq 1$, then there exists a continuum of type I equilibria, each with zero limit capital per capita.*

(ii) If $\frac{1+n}{\eta} < \min\{1, \nu\}$, then there exists a unique type II equilibrium.

(iii) If $\nu \leq \frac{1+n}{\eta} < 1$, then if

$$\alpha = \alpha^*, \quad \frac{1+n}{\eta} \leq -\frac{\sigma\rho}{1+\rho}, \tag{20}$$

or

$$\alpha < \alpha^*, \quad \nu = \frac{1+n}{\eta} > -\frac{\sigma\rho}{1+\rho}, \tag{21}$$

then there exists a unique type III equilibrium with harvesting speed indicator θ; if

$$\alpha < \alpha^*, \quad \nu \leq \frac{1+n}{\eta} < -\frac{\sigma\rho}{1+\rho}, \tag{22}$$

then there exists a continuum of type III equilibria. More precisely, there is a $\epsilon^* \in [\epsilon_*, \theta)$ such that for any $\epsilon \in [\epsilon^*, \theta]$, there is a type III equilibrium with harvesting speed indicator ϵ. In all other cases, there is no equilibrium.

Concerning the efficiency of equilibrium, we have the following.

PROPOSITION 9. *(i) If $\eta \leq 1 + n$, then any type I equilibrium is Pareto efficient.*

(ii) If $\nu \leq \frac{1+n}{\eta} < 1$, then there are three cases.

First, if (20) holds, then the unique type III equilibrium is Pareto efficient if $-\sigma < \frac{1+\rho}{\rho}$; it is Pareto inefficient if $-\sigma > \frac{1+\rho}{\rho}$.

Second, if (21) holds, then the unique type III equilibrium is Pareto efficient.

Third, if (22) holds, then there exists $\overline{\epsilon} \in [\epsilon^*, \theta]$ such that the type III equilibrium with harvesting speed indicator below $\overline{\epsilon}$ is Pareto efficient; whereas any type III equilibrium with harvesting speed indicator above $\overline{\epsilon}$ is Pareto inefficient.

Remark 16. When $\sigma < 0$, the natural resource is essential for production, in the sense that output drops to zero in its absence.

If the regeneration capacity of the resource is weak compared to the population growth, the resource stock per capita will tend to zero, making overaccumulation of the resource impossible. Additionally, because the factors are complementary, capital and the resource are closely linked in production, preventing over-accumulation of capital and thereby ensuring the Pareto efficiency of the economy.

If the regeneration capacity of the resource is strong compared to the population growth and there is no steady state, the resource stock per capita will tend to infinity. And, in general, similar to the case when $\sigma \in (0,1)$, there exists a threshold for the harvesting speed indicator, below which the economy is Pareto efficient, and above which it becomes inefficient. In other words, the slower the resource extraction, the higher the likelihood that the economy will be Pareto efficient. And this threshold, in principle, is concerned with the marginal regeneration capacity.

Remark 17. As to the existence of equilibrium, typically when $\nu < \frac{1+n}{\eta} < 1$ and $\alpha > \alpha^*$, there is no equilibrium. This suggests that in a typical scenario where the regeneration capacity of the resource is very strong and the capital share excessively large, but the resource and the capital are complementary in production, they cannot maintain a coherent relationship, preventing the economy from following an equilibrium path.

5.2.3. Comparison Between Different σ

We know that $\sigma \in (0,1)$ indicates substitutability between the factors, $\sigma < 0$ indicates complementarity between the factors, and $\sigma = 0$ represents the midpoint between the two.

Here, we compare the behavior of the dynamical system $\mathscr{D}$ under different values of σ, including $\sigma = 0$. Specifically, we examine how varying σ influences the system's trajectories, equilibrium types, and overall economic efficiency. By contrasting these cases, we gain insight into the role that σ plays in determining the stability and optimality of the dynamical system.

In the Cobb–Douglas case where $\sigma = 0$, the dynamical system $\mathscr{D}$ simplifies to[21]

$$
\begin{aligned}
k_{t+1} &= \frac{1}{1+n} k_t^{\alpha} z_t^{\gamma-1} \left[(\theta+\gamma) z_t - \gamma s_t\right], \\
s_{t+1} &= \frac{\eta}{1+n}(s_t - z_t), \\
z_{t+1} &= \frac{\eta}{\alpha(1+n)} \left[(\theta+\gamma) z_t - \gamma s_t\right].
\end{aligned}
$$

[21] See also (12), (13) and (14).

From the analysis of the Cobb–Douglas case in Section 5.1, we know that when $\sigma = 0$, this dynamical system has a unique positive solution, and the economy possesses a unique equilibrium, which can be of type I, type II, or type III, depending on whether $\eta\delta <, =, > 1 + n$, respectively, and it is socially optimal with respect to W_δ, and therefore Pareto efficient.

For the dynamical system $\mathscr{D}$, including the case $\sigma = 0$, the system behavior changes as σ varies. Depending on other parameters, the system may exhibit continuity in some cases, whereas in others, bifurcation may occur.

First, for the nontrivial steady state: when $\sigma \neq 0$, it exists if and only if condition (18) holds; when $\sigma = 0$, by Lemma 5, it exists if and only if $\eta\delta = 1 + n$. For $\sigma = 0$, (18) reduces to

$$\frac{1+n}{\eta} < \min\left\{1, \frac{\theta}{\beta}\frac{\beta+\gamma}{\alpha}\right\},$$

which is satisfied naturally when $\eta\delta = 1 + n$. Therefore, regarding the nontrivial steady state, the dynamical system $\mathscr{D}$ exhibits continuity with respect to σ.

In the sequel, we present some examples to further illustrate the role of σ in the dynamical system's behavior.

Example 1. Suppose $\alpha < 1/(2e)$, and

$$\frac{\rho}{1+\rho}\frac{\beta+\gamma}{\alpha} < \frac{1+n}{\eta} < \delta.$$

For $\sigma = 0$, because $\eta\delta > 1 + n$, then there exists a unique trajectory of $\mathscr{D}$, along which each of k_t, s_t, z_t tends to infinity, all other trajectories lead to system collapse within a finite time, and correspondingly, there is a unique equilibrium of type III, which is of course Pareto efficient.

For $\sigma > 0$ near $\sigma = 0$ locally, because $\nu < \frac{1+n}{\eta} < 1$, there exists a unique trajectory of $\mathscr{D}$, along which (k_t, s_t, z_t) tends to $(\pi^{-1}(\theta), 0, 0)$, all other trajectories lead to system collapse within a finite time, and correspondingly, there is a unique equilibrium of type I, which is Pareto efficient.

For $\sigma < 0$ near $\sigma = 0$ locally, we have $\nu < \frac{1+n}{\eta} < 1$. In addition, locally near $\sigma = 0$, α^* is sufficiently close to $1/e$, and thus, locally near $\sigma = 0$, we have $\alpha < \alpha^*$. But, locally near $\sigma = 0$, it does not hold that

$$\frac{1+n}{\eta} > -\frac{\sigma\rho}{1+\rho}.$$

Therefore, by Proposition 8, there is no equilibrium. All trajectories of $\mathscr{D}$ lead to the collapse of the system in a finite time.

Thus, in this case, bifurcation occurs at $\sigma = 0$.

Example 2. Suppose $\eta < 1 + n$. Then, locally near $\sigma = 0$, the system exhibits continuity. In fact, regardless of the value of σ, $\mathscr{D}$ has trajectories (possibly unique) converging to $(k, 0, 0)$ for some (different) k, all other trajectories exit the region Θ within a finite time. Correspondingly, for any σ, the economy consistently exhibits type I equilibria. The efficiency of the equilibria manifests differently: for $\sigma \in (0, 1)$, there exists a threshold for the harvesting speed indicator, below which the economy is Pareto efficient, and above which it becomes inefficient; for $\sigma \leq 0$, all equilibria are Pareto efficient.

To sum up, roughly, if $\eta > 1+n$, then the system may exhibit bifurcation at $\sigma = 0$; on the contrary, if $\eta < 1 + n$, then the system exhibits continuity at $\sigma = 0$.

Additionally, in general, in most cases, the economy exhibits multiple equilibria. However, in the Cobb-Douglas case, where all variables are in fixed proportion, the set of equilibria reduces to a singleton.

The primary distinction between the cases $\sigma > 0$ and $\sigma < 0$ lies in the behavior of the resource stock per capita. When $\sigma > 0$, the resource stock per capita cannot tend to infinity and thus there is no type III equilibrium, even if the resource regeneration capacity is very large. However, when $\sigma < 0$, this becomes possible. Additionally, when $\sigma < 0$, the possibility of nonequilibrium arises.

A key commonality between the cases $\sigma \in (0, 1)$ and $\sigma < 0$ is that, roughly speaking, the slower the resource extraction, the higher the likelihood that the economy will be Pareto efficient. This principle holds regardless of whether the factors are substitutable or complementary. But, more precisely, this principle applies to the case $\eta < 1 + n$ when $\sigma \in (0, 1)$ (the resource stock per capita tends to zero) and to the case $\eta > 1 + n$ when $\sigma < 0$ (the resource stock per capita tends to infinity). The intuition behind this principle, within the framework of general CES technology, is that whether the factors are substitutable or complementary, they remain interconnected through a weak proportional relationship. Faster resource extraction leads to greater resource use in production, which increases the demand for capital in the production process. Over time, this results in higher capital accumulation, which raises the likelihood of capital overaccumulation, potentially leading to inefficiency.

5.2.4. Comparison with Classical Diamond OLG Model

Our model in the general CES form reduces to the classical Diamond OLG model[22] without natural resources when $\gamma = 0$.

It is easy to see that in this Diamond model, there exists a unique equilibrium, in which for any $t \in \mathbb{N}$,

$$\begin{aligned} k_{t+1} &= \frac{\theta}{1+n} \left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}, \\ \omega_t &= \beta \left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}, \\ 1 + r_t &= \alpha k_t^{\sigma-1} \left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}. \end{aligned}$$

First, we consider the case, where $\sigma \in (0, 1)$. It is easy to see that $\lim_{t\to\infty} k_t = \hat{k}$, where $\hat{k}$ is determined uniquely by

$$\hat{k} = \frac{\theta}{1+n} \left(\alpha \hat{k}^\sigma + \beta\right)^{(1-\sigma)/\sigma}.$$

And hence,

$$\lim_{t\to\infty} (1 + r_t) = (1 + n)\frac{\alpha}{\theta}\hat{k}^\sigma.$$

Therefore, by Corollary 4, the equilibrium is Pareto efficient (respectively Pareto inefficient), if

$$\hat{k}^\sigma > \frac{\theta}{\alpha} \quad (\text{resp. } \hat{k}^\sigma < \frac{\theta}{\alpha}),$$

or, equivalently,

$$n < \hat{n} \quad (\text{resp. } n > \hat{n}),$$

where

$$\hat{n} = \left[\alpha \left(2 + \rho^{-1}\right)^{1-\sigma}\right]^{1/\sigma} - 1.$$

Thus, to obtain efficiency, the population growth rate must not be too high, nor may the capital share be too low. Relative to the OLG economy in Section 5.2.1, Proposition 7 shows that even when the population growth rate is high, introducing a natural resource can render the economy Pareto efficient—at least along some equilibrium paths in the presence of multiple equilibria. Hence, the natural resource serves to promote efficiency.

[22]The "Diamond model" here is CES. Recall that the Diamond model in Remark 12 is Cobb–Douglas.

In the sequel, we only consider the case where $\sigma < 0$. Define

$$n^* := \frac{\rho\left[\alpha(1-\sigma)\right]^{1/\sigma}}{\left(1-\sigma^{-1}\right)(1+\rho)} - 1, \quad n_* := \left[\alpha\left(1+\frac{1+\rho}{\rho}\right)^{1-\sigma}\right]^{1/\sigma} - 1.$$

One can directly check that $-1 < n_* < n^*$.

It is easy to verify that if $n > n^*$, then $\lim_{t\to\infty} k_t = 0$. Therefore, as $t \to \infty$,

$$\frac{1}{1+r_{t+1}}\frac{\omega_{t+1}N_{t+1}}{\omega_t N_t} = \frac{1+n}{\alpha}\frac{k_{t+1}^{1-\sigma}}{\left(\alpha k_t^\sigma + \beta\right)^{(1-\sigma)/\sigma}} = \frac{\theta}{\alpha}k_{t+1}^{-\sigma} \to 0,$$

which implies $\lim_{t\to\infty} D_t\omega_t N_t = 0$. Then, by Theorem 3, the equilibrium is Pareto efficient.

In the following, suppose $n < n^*$. Then, the limit capital per capita k and the limit interest rate r satisfy

$$\frac{1+r}{1+n} = x = \frac{\alpha}{\theta}k^\sigma,$$

and x is the smaller of the two positive roots of the equation.

$$(1+n)^\sigma x = \alpha\left(x + \beta/\theta\right)^{1-\sigma}.$$

If $n \in (n_*, n^*)$, then one can check that $x > 1$. Therefore, $r > n$. In addition, the limit wage is positive. Therefore, by Corollary 4, the equilibrium is Pareto efficient.

If $n < n_*$, then one can check that $x < 1$. Therefore, $r < n$. In addition, the limit wage is positive. Therefore, by Corollary 4, the equilibrium is Pareto inefficient.

However, in the case $n < n_*$, when natural resources are introduced into the economy, as in our CES model, the economy becomes Pareto efficient, provided that $\eta < 1 + n$.

In summary, regarding the effect of introducing natural resources on economic efficiency, we find that, roughly speaking, substitutability implies that stronger resource regeneration increases the likelihood of efficiency, whereas complementarity implies that weaker regeneration does so. In both cases, the mechanism is to mitigate capital overaccumulation.

At the end of this subsection, we point out that the sustainability issue in this CES model is straightforward, and thus omitted from the discussion.

5.3. Quadratic Regeneration Function

Assume $G(x) = \lambda x(1 - x/B)$, and for any $t \in \mathbb{N}$,

$$F^t(K, L, R) = A_t K^\alpha L^\beta R^\gamma, \quad N_t = (1+n)^t, \quad A_t = (1+g)^t,$$

where $n \geq 0, g \geq 0,\ \lambda > 0, B > \overline{S}_0$, and $\rho, \alpha, \beta, \gamma \in (0, 1)$ are constants, satisfying $\alpha + \beta + \gamma = 1$.

And assume λ and B are sufficiently large. Here, λ is the intrinsic growth rate of the natural resource, and B is the environmental carrying capacity for this natural resource.

As mentioned in the literature review, Krautkraemer (1999), in an OLG economy with a natural resource but no physical capital, suggests that when the resource's output share is relatively small, steady-state equilibrium is Pareto inefficient, but does not specify how small it must be. Here, for an economy with capital, we provide a similar but more precise result regarding the dynamic equilibrium.

5.3.1. Existence and Uniqueness of Equilibrium

It is easy to verify that an equilibrium exists if and only if the following difference dynamical system for $(K_t, S_t, R_t)_{t\in\mathbb{N}}$ with $K_0 = \overline{K}_0$ and $S_0 = \overline{S}_0$ has a positive solution:

$$K_{t+1} = \frac{Y_t}{1+\rho}\left[\rho\beta - \left(\frac{G(S_t - R_t)}{G'(S_t - R_t)} + \rho(S_t - R_t)\right)\frac{\gamma}{R_t}\right], \tag{23}$$

$$S_{t+1} = G(S_t - R_t), \tag{24}$$

$$R_{t+1} = \frac{R_t G'(S_t - R_t)}{\alpha(1+\rho)}\left[\rho\beta - \left(\frac{G(S_t - R_t)}{G'(S_t - R_t)} + \rho(S_t - R_t)\right)\frac{\gamma}{R_t}\right], \tag{25}$$

which, in turn, if and only if the planar dynamical system for $(S_t, R_t)_{t\in\mathbb{N}}$, described by (24) and (25), with $S_0 = \overline{S}_0$, has a positive solution.

This planar dynamical system has two steady states: $(0, 0)$ and (S^*, R^*), satisfying

$$R^* = G(x^*) - x^*, \quad S^* = G(x^*),$$

where $x^* \in (0, B/2)$ is determined uniquely by

$$\lambda\left(\alpha + \frac{\gamma}{1+\rho}\right)\left(1 - \frac{x^*}{B}\right) - \alpha = \frac{\rho\beta\lambda}{1+\rho}\left(1 - \frac{x^*}{B/2}\right)\left[\lambda\left(1 - \frac{x^*}{B}\right) - 1 - \frac{\gamma}{\beta}\right].$$

By the eigenvalue method[23], one can see that $(0, 0)$ is a source, to which no feasible path converges; (S^*, R^*) is a saddle point, to which a unique saddle path converges.

Therefore, there exists a unique $R_0 > 0$ which induces a unique path converging to this saddle. The unique equilibrium then follows.

Consequently, we obtain

Proposition 10. *The equilibrium exists and is unique, and the corresponding path of $(S_t, R_t)_{t\in\mathbb{N}}$ converges to a saddle.*

5.3.2. Pareto Efficiency

Let

$$\kappa := \frac{1+\rho}{\rho}\alpha + \left(\frac{1}{\rho} + \frac{2(1+\rho)}{\rho(\lambda-1)}\right)\gamma,$$

which represents a weighted sum of the two capital shares: α and γ, and hence, can be referred to as a composite capital index. Note that this index only concerns the intrinsic growth rate of the natural resource, not the carrying capacity.

Proposition 11. [24] *The equilibrium is Pareto efficient if $\beta < \kappa$; it is Pareto inefficient if $\beta > \kappa$.*

Remark 18. The relative magnitude of the labor share plays a crucial role. If the labor share is less than the composite capital index, the equilibrium is efficient; however, if the labor share exceeds the composite capital index, the equilibrium becomes inefficient.

On the simplex $\mathcal{A} = \{(\alpha, \beta, \gamma) | \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \in [0, 1]\}$, the line segment $\beta = \kappa$ has two endpoints, the coordinates of which are $(\overline{\alpha}, \overline{\beta}, 0)$ and $(0, \underline{\beta}, \underline{\gamma})$, respectively, where

$$\overline{\alpha} := \frac{\rho}{1+2\rho}, \quad \overline{\beta} := \frac{1+\rho}{1+2\rho}, \quad \underline{\beta} := \frac{1+\lambda+2\rho}{(1+\rho)(1+\lambda)}, \quad \underline{\gamma} := \frac{\rho(\lambda-1)}{(1+\rho)(1+\lambda)}.$$

Recall λ is sufficiently large, then $\underline{\beta} < \overline{\beta}$.

[23]The Jacobian matrix at the steady state $(0, 0)$ has two eigenvalues bigger than 1. Therefore, the steady state $(0, 0)$ is a source. The Jacobian matrix at the steady state (S^*, R^*) has two positive eigenvalues: one is smaller than 1, the other is greater than 1. Therefore, (S^*, R^*) is a saddle.

[24]If $\gamma = 0$, then this result coincides with that in the classical Diamond OLG model.

Clearly, if $\beta < \underline{\beta}$, then $\beta < \kappa$; if $\beta > \underline{\beta}$, then $\beta > \kappa$. And, of course, if $\gamma > \underline{\gamma}$, then $\beta < \underline{\beta}$. Then, from Proposition 9, we can easily have the following corollary.

COROLLARY 5. *If $\beta < \underline{\beta}$, the equilibrium is Pareto efficient; if $\beta > \overline{\beta}$, the equilibrium is Pareto inefficient.*

From this corollary, we can say roughly that if the technology is capital-intensive (either the physical capital or the natural capital), then the economy is efficient; on the contrary, if the technology is labor-intensive, then the economy is inefficient.

In particular, suppose there is neither technical growth nor population growth. Then K_t will converge to some $K^* > 0$. If $\beta = \kappa$, $K^* = K_{GR}$, where K_{GR} is the so-called Golden Rule level of capital; if $\beta < \kappa$, $K^* < K_{GR}$, and the economy is efficient; if $\beta > \kappa$, then $K^* > K_{GR}$, indicating capital overaccumulation, and the economy is inefficient.

5.3.3. Sustainability

It follows from (23) that

$$y_{t+1} \sim \left(\frac{1+g}{(1+n)^\gamma}\right)^t y_t^\alpha,$$

where $y_y = Y_t/N_t$ is the output per capita. Based on this, we can immediately derive the following result.

PROPOSITION 12. *If $1+g < (1+n)^\gamma$, the economy contracts; if $1+g = (1+n)^\gamma$, the economy is sustainable in the long run; if $1+g > (1+n)^\gamma$, the economy grows without bound.*

Remark 19. Whether the economy contracts depends solely on the rate of technical progress, population growth, and the resource share. It is independent of the distribution between capital and labor shares.

6. EXTENSION AND FURTHER DISCUSSION

6.1. Multiple Resources

The main results can be easily extended to the case of multiple natural resources, where the regeneration capacities are independent of each other. In other words, no cross-effects are present in their regeneration. More specifically, the scenario is as follows.

Consider multiple types of natural resources, labeled type-1, type-2,…,type-J, where J is some natural number.

For any $j = 1, \dots, J$, the regeneration function of type-j resource is G_j, being smooth, concave, and nonnegative, defined on $[0, \infty)$, with properties $G_j(0) = 0$, $G'_j(0) \in (0, \infty]$, $G'_j(x) > 0$, $\forall x > 0$.

The dynamics of type-j resource are

$$S^j_{t+1} = G_j(S^j_t - R^j_t),$$

where S^j and R^j represent the stock and extraction of type-j resource, respectively.

In this context, the total value of assets at time t becomes

$$V_t = (1 + r_t)K_t + \sum_{j=1}^{J} p^j_t S^j_t,$$

where p^j_t is the price of type-j resource at time t. The generalized Hotelling rule holds for every type of resources. That is, for any type-j resource, we have

$$\frac{p^j_{t+1} G'_j(S^j_t - R^j_t)}{p^j_t} = 1 + r_{t+1}, \quad \forall t \in \mathbb{N}.$$

In this case, the assumption **A2** "the resource is important in production relative to labor" should be modified to the condition "at least one of the resources is important in production relative to labor."

6.2. OLG with Land

Regarding the OLG economy with land, similar results hold true. Now, the production function is $F^t(K, L, X)$, where X is the input of land.

In an equilibrium, let p_t and q_t be the corresponding price of land and the rental of land at time t, respectively.

The no-arbitrage condition implies that for any $t \in \mathbb{N}$,

$$p_t = \frac{p_{t+1} + q_{t+1}}{1 + r_{t+1}},$$

then, $D_t p_t = D_{t+1} p_{t+1} + D_{t+1} q_{t+1}$. Therefore, $\lim_{t \to \infty} D_t q_t = 0$, and $(D_t p_t)_{t \in \mathbb{N}}$ is decreasing, and hence, there is $\beta_0 \geq 0$ such that $\lim_{t \to \infty} D_t p_t = \beta_0$, and for any $t \in \mathbb{N}$, $p_t = f_t + \beta_t$, where

$$f_t = \frac{1}{D_t} \sum_{s=t+1}^{\infty} D_s q_s, \quad \beta_t = \frac{\beta_0}{D_t},$$

are the fundamental and the bubble of land at t, respectively.

The total value of assets is $V_t = (1 + r_t)K_t + p_t + q_t$, assuming that the quantity of land is one unit. And the total income I_t coincides with wages $\omega_t N_t$.

Assumption **A1** is not needed. In principle, Theorems 1–5 still hold. As a corollary of Theorem 3, if

$$\limsup_{t\to\infty} \frac{q_t}{\omega_t N_t} > 0, \tag{26}$$

that is, land is important in production relative to labor, then $\varliminf_{t\to\infty} D_t \omega_t N_t = 0$. Therefore, the equilibrium is Pareto efficient.

Condition (26) is a bit weaker than the condition

$$\liminf_{t\to\infty} \frac{q_t}{Y_t} > 0,$$

which is used in Proposition 1 in Rhee (1991) to guarantee the Pareto efficiency of the equilibrium.

6.3. OLG without Capital

An economy without capital can be viewed as a special case of the general economy with capital, as discussed in Section 3. In this scenario, capital remains at zero throughout, including the initial capital endowment of ancestors. However, the interest rate at any time $t \geq 1$ still exists as a reference for agents borrowing or lending at time $t - 1$, though in equilibrium, the quantity of borrowing or lending is zero. In the definition of equilibrium, the interest rate at time $t = 0$ can be ignored because it has no impact on anyone. The generalized Hotelling rule (1) still holds.

From this perspective, all the main results from Sections 3 and 4 continue to hold, except for Theorem 4, where the growth rate of capital is not well defined and thus does not apply in this case.

As an application of Theorem 3, consider an example as follows. Suppose the production functions are of the form: $F^t(L, R) = A_t L f(R/L)$, where $A_t > 0$ is a constant and f is smooth, concave, and satisfying $f(0) = 0$, $f'(0+) \in (0, \infty]$, and the regeneration function is linear: $G(x) = \eta x$, where $\eta > 0$ is a constant. The utility function satisfies the standard conditions such as being smooth, concave, and meeting Inada conditions, etc. Then, analogously to Olson and Knapp (1997), one can show that the equilibrium exists (possibly multiple).

Take anyone of the equilibria. Since $\eta^{-(t+1)} S_{t+1} = \eta^{-t}(S_t - R_t)$, we have $\sum_{t=0}^{\infty} \eta^{-t} R_t \leq S_0$, therefore $\lim_{t\to\infty} \eta^{-t} R_t = 0$.

Denote the resource extraction and resource stock per capita as $z_t =: R_t/N_t$ and $s_t =: S_t/N_t$, respectively.

Suppose

$$0 < \varliminf_{t\to\infty} \eta^{-t} N_t \le \varlimsup_{t\to\infty} \eta^{-t} N_t < \infty. \tag{27}$$

Then, $\lim\limits_{t\to\infty} z_t = 0$.

In addition, since $F^t(N_t, R_t) = N_t a_t + N_{t-1} b_t \ge N_{t-1} b_t = p_t S_t$, and $p_t = A_t f'(z_t)$, then $f(z_t) \ge f'(z_t) s_t$. By letting $t \to \infty$, and noticing $f(0) = 0$, $f'(0+) \in (0, \infty]$, we obtain $\lim\limits_{t\to\infty} s_t = 0$. That is, along any equilibrium path, the resource stock per capita converges to zero.

Now we consider the efficiency. By the generalized Hotelling rule, for any $t \in \mathbb{N}$, $1 + r_{t+1} = \eta p_{t+1}/p_t$, then $D_t = \eta^{-t} p_0/p_t$. Therefore, as $t \to \infty$,

$$D_t Y_t = \eta^{-t} \frac{p_0}{A_t f'(z_t)} A_t N_t f(z_t) = p_0 \eta^{-t} N_t \frac{f(z_t)}{f'(z_t)} \to 0,$$

which yields $D_t \omega_t N_t \to 0$. Then, by Theorem 3, the equilibrium is Pareto efficient.

In some special cases, the condition (27) is superfluous. For example, consider a special case of the example in section 5.1, where $\alpha = 0$, $\overline{K}_0 = 0$. In this case, the production function at time t is $F^t(L, R) = A_t L^\beta R^\gamma$.

In this case, Propositions 1, 2, and 3 still hold. There exists a unique equilibrium that is Pareto efficient. At equilibrium, for any $t \in \mathbb{N}$,

$$N_t a_t = \frac{\beta}{1+\rho} Y_t, \quad N_{t-1} b_t = (\theta + \gamma) Y_t, \quad R_t = \tau S_t,$$

$$1 + r_{t+1} = \frac{Y_{t+1}}{\delta Y_t},$$

where $Y_t = A_t N_t^\beta R_t^\gamma$, $\tau = \gamma/(\theta + \gamma)$, $\delta = 1 - \tau$, $\theta = \rho\beta/(1 + \rho)$.

And, under assumption **A4**, the equilibrium allocation is socially optimal with respect to the social welfare functional W_δ. That is, the equilibrium allocation is the solution of the following social planner's problem:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \delta^t \left(\delta \ln a_t + \rho \ln b_t \right), \\ \text{s.t} \quad & S_{t+1} = \eta (S_t - R_t), \\ & N_t a_t + N_{t-1} b_t \le A_t N_t^\beta R_t^\gamma. \end{aligned}$$

According to Olson and Knapp (1997, p.290): "OLG equilibria differ substantially from the outcome under a social planning exercise, and there does not exist a definitive relation between extractions and prices in the two cases". However, this assertion is incorrect. While the equilibrium allocation may not be socially optimal with respect to W_ρ, which is considered in Olson and Knapp (1997), it is socially optimal with respect to W_δ. When evaluating social optimality, the individual discount rate should be replaced by the social discount rate, which is embedded in the market system.

7. CONCLUSION

In this paper, we consider a two-period OLG model with three factors of production: physical capital, labor, and natural resources. We discuss the issue of Pareto efficiency of the equilibrium allocation.

Our main contribution to the literature is that we present general sufficiency conditions and general necessary conditions for the Pareto efficiency of the equilibrium allocation in the OLG economies with natural resources and physical capital. In principle, we compare the growth rates of capital, income, or total asset value with the interest rate. Our findings suggest that, broadly speaking, if any of these growth rates is lower than the interest rate, the equilibrium is efficient. Conversely, if any of these growth rates surpasses the interest rate, the equilibrium becomes inefficient.

A secondary contribution is the finding that, in the case where the resource regeneration function is linear and the production function follows a CES form beyond Cobb–Douglas, there is generally a threshold for the resource harvesting speed. If the harvesting speed is below this threshold, the economy operates efficiently; if it exceeds the threshold, inefficiency arises.

Another contribution is for the case where the resource regeneration function is quadratic. We provide a precise composite capital index and demonstrate that if the labor share is below this index, the economy operates efficiently, whereas if the labor share exceeds this index, the economy becomes inefficient.

Moreover, our findings suggest that under certain conditions, natural resources can enhance economic efficiency, largely through their interaction with capital. This highlights the potential role of resource management in improving economic outcomes.

While our results offer important insights, they remain incomplete. We have not provided general necessary and sufficient conditions for Pareto efficiency of equilibrium, leaving this as an open problem for future research.

Another promising direction for future work is to explore stochastic OLG models that account for uncertainties arising from the random variability of natural resources and environmental conditions. Furthermore, examining government or institutional interventions could be critical, particularly in cases where resource use leads to pollution that exacerbates market inefficiencies.

APPENDIX A

We need the following three lemmas. The proofs of Lemma 3 and Lemma 4 are straightforward and hence omitted. One proof of Lemma 5 can be found in Mourmouras (1991) or Farmer et al. (2010), which uses the eigenvalue method in the planar difference dynamical system. Here, we present another proof, which has its own interest.

LEMMA 3. *A program $\{C_t^*, K_t^*, S_t^*, R_t^*\}_{t\in\mathbb{N}}$ is dynamically efficient if and only if for any program $\{C_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$,*

$$C_t \geq C_t^*, \quad \forall t \geq 1$$

implies

$$C_0^* \geq C_0.$$

LEMMA 4. *An allocation $\{a_t^*, b_t^*, K_t^*, S_t^*, R_t^*\}_{t\in\mathbb{N}}$ is Pareto efficient if and only if for any allocation $\{a_t, b_t, K_t, S_t, R_t\}_{t\in\mathbb{N}}$,*

$$U(a_t, b_{t+1}) \geq U(a_t^*, b_{t+1}^*), \quad \forall t \geq 0$$

implies

$$b_0^* \geq b_0.$$

LEMMA 5. *The following two statements about $\{S_t, R_t\}_{t\in\mathbb{N}}$ with given $S_0 > 0$ are equivalent:*

(I) for any $t \in \mathbb{N}$,

$$S_{t+1} = \eta(S_t - R_t) \geq 0,$$
$$R_{t+1} = \frac{\eta}{\alpha}\left[(\theta+\gamma)R_t - \gamma S_t\right] \geq 0;$$

(II) for any $t \in \mathbb{N}$,

$$S_t = (\eta\delta)^t S_0,$$
$$R_t = \tau(\eta\delta)^t S_0,$$

where δ, τ are defined in the beginning of Section 5.1.

Proof. One can easily verify that (II) implies (I). In the sequel, we prove that (I) implies (II). First of all, we show that for any $t \in \mathbb{N}$ and any $n \in \mathbb{N}$, it holds that

$$x_n S_t \leq R_t \leq y_n S_t, \tag{A.1}$$

where

$$x_{n+1} = \frac{\gamma + \alpha x_n}{\theta + \gamma + \alpha x_n}, \quad x_0 = 0,$$

$$y_{n+1} = \frac{\gamma + \alpha y_n}{\theta + \gamma + \alpha y_n}, \quad y_0 = 1.$$

We prove (A.1) by use of the method of mathematical induction with respect to n. First, obviously, (A.1) holds for $n = 0$ and any $t \in \mathbb{N}$. Now, suppose that (A.1) holds for n and any $t \in \mathbb{N}$. Then, for any $t \in \mathbb{N}$, notice that (A.1) holds for n and $t + 1$, that is,

$$x_n S_{t+1} \leq R_{t+1} \leq y_n S_{t+1},$$

which is equivalent to

$$x_{n+1} S_t \leq R_t \leq y_{n+1} S_t,$$

and hence, (A.1) also holds for $n + 1$ and any $t \in \mathbb{N}$. It follows that (A.1) holds for any $t \in \mathbb{N}$ and any $n \in \mathbb{N}$.

Next, clearly, $\{x_n\}_{n\in\mathbb{N}}$ is increasing and bounded above, and $\{y_n\}_{n\in\mathbb{N}}$ is decreasing and bounded below, and hence, each of these two sequences has a limit. Let $\lim_{t\to\infty} x_t = x$, $\lim_{t\to\infty} y_t = y$. Then, $x, y \in (0,1)$ and satisfy

$$x = \frac{\gamma + \alpha x}{\theta + \gamma + \alpha x}, \quad y = \frac{\gamma + \alpha y}{\theta + \gamma + \alpha y},$$

which implies $x = y = \tau$. Consequently, $R_t = \tau S_t$, $\forall t \in \mathbb{N}$, which yields (I) immediately. ▌

Proof of Theorem 1. By Lemma 3, it suffices to show that for any program $\{C'_t, K'_t, S'_t, R'_t\}_{t\in\mathbb{N}}$, if $C'_t \geq C_t$, $\forall t \geq 1$, then $C'_0 \leq C_0$.

In fact, taking large $T \in \mathbb{N}$ arbitrarily, we have

$$\begin{aligned}
& C'_0 - C_0 \\
\leq\ & \sum_{t=0}^{T-1} D_t \left[\left(F^t(K'_t, N_t, R'_t) - K'_{t+1}\right) - \left(F^t(K_t, N_t, R_t) - K_{t+1}\right)\right] \\
& + \sum_{t=0}^{T-1} D_{t+1}p_{t+1}\left[\left(G(S'_t - R'_t) - S'_{t+1}\right) - \left(G(S_t - R_t) - S_{t+1}\right)\right] \\
\leq\ & \sum_{t=0}^{T-1} D_t \left[(1 + r_t)(K'_t - K_t) + p_t(R'_t - R_t) - (K'_{t+1} - K_{t+1})\right] \\
& + \sum_{t=0}^{T-1} D_{t+1}p_{t+1}\left\{G'(S_t - R_t)\left[(S'_t - S_t) - (R'_t - R_t)\right] - \left(S'_{t+1} - S_{t+1}\right)\right\} \\
=\ & \sum_{t=0}^{T-1} \left[D_{t-1}(K'_t - K_t) + D_t p_t(R'_t - R_t)\right] - \sum_{t=1}^{T} D_{t-1}(K'_t - K_t) \\
& + \sum_{t=0}^{T-1} D_t p_t\left[(S'_t - S_t) - (R'_t - R_t)\right] - \sum_{t=1}^{T} D_t p_t\left(S'_t - S_t\right) \\
\leq\ & D_T V_T.
\end{aligned}$$

Letting $T \to \infty$ (along some subsequence of natural numbers), we obtain $C'_0 \leq C_0$. ▌

Proof of Theorem 2. Suppose $G(x) = \eta x$, where $\eta > 0$ is some constant. Then $\eta^{-(t+1)}S_{t+1} = \eta^{-t}(S_t - R_t)$, $\forall t \in \mathbb{N}$. Therefore,

$$\sum_{t=0}^{\infty} \eta^{-t} R_t \leq S_0,$$

and the sequence $(\eta^{-t}S_t)_{t\in\mathbb{N}}$ is strictly decreasing and hence converges to some nonnegative number θ.

It must hold that $\theta = 0$. Otherwise, the extra amount of resource can be used in any period of time and produce more consumption goods, which can be distributed to the people in that period, and all the other periods are not affected. Then the aggregate consumption in that period is increased strictly, and the aggregate consumptions at any other times are not changed. This contradicts the dynamic efficiency of the equilibrium allocation. Thus, $\theta = 0$.

The generalized Hotelling rule implies that $D_t p_t = \eta^{-t} p_0$, $\forall t \in \mathbb{N}$. Therefore,

$$\lim_{t\to\infty} D_t p_t S_t = 0.$$

It is left to show $\lim_{t\to\infty} D_t K_{t+1} = 0$. We know that for any $t \in \mathbb{N}$,

$$C_t + K_{t+1} = F^t(K_t, N_t, R_t) = (1 + r_t)K_t + \omega_t N_t + p_t R_t,$$

where $C_t = N_t a_t + N_{t-1} b_t$. Then,

$$D_t C_t + D_t K_{t+1} = D_{t-1} K_t + D_t \omega_t N_t + D_t p_t R_t.$$

Therefore,

$$\sum_{s=0}^{t} D_s C_s + D_t K_{t+1} = D_{-1} K_0 + \sum_{s=0}^{t} D_s \omega_s N_s + p_0 \left(\sum_{s=0}^{t} \eta^{-s} R_s \right). \quad \text{(A.2)}$$

Since

$$\sum_{s=0}^{\infty} \eta^{-s} R_s = S_0,$$

and noticing assumption **A2**, we know that

$$\sum_{s=0}^{\infty} D_s \omega_s N_s < \infty,$$

and hence, by (A.2),

$$\sum_{s=0}^{\infty} D_s C_s < \infty.$$

Then, once again, by (A.2), we get that $\lim_{t\to\infty} D_t K_{t+1}$ exists. We prove that this limit is 0. Suppose not. Then there exist $\varepsilon > 0$ and $T \in \mathbb{N}$ such that for any $t \geq T$,

$$\sum_{s=t}^{\infty} D_s C_s < \varepsilon < D_t K_{t+1}.$$

For any $t \geq T$, let

$$\lambda_t = \frac{1}{\varepsilon} \sum_{s=t}^{\infty} D_s C_s \in (0, 1).$$

Then, for any $t \geq T$, we have

$$(\lambda_t - \lambda_{t+1}) D_t K_{t+1} > D_t C_t.$$

Now, for any $t \geq T$, let

$$K'_t = \lambda_t K_t, \quad R'_t = \lambda_t R_t, \quad C'_t = F^t(K'_t, N_t, R'_t) - K'_{t+1}.$$

We have that for any $t \geq T$,

$$\begin{aligned} D_t C'_t &= D_t F^t(K'_t, N_t, R'_t) - D_t K'_{t+1} \\ &\geq \lambda_t D_t F^t(K_t, N_t, R_t) - \lambda_{t+1} D_t K_{t+1} \\ &\geq (\lambda_t - \lambda_{t+1}) D_t K_{t+1} > D_t C_t. \end{aligned}$$

Thus, $C'_t > C_t, \quad \forall t \geq T$.

Now construct a program $(C'_t, K'_t, S'_t, R'_t)_{t \in \mathbb{N}}$ as follows: for any $t < T$, let

$$(C'_t, K'_t, S'_t, R'_t) = (C_t, K_t, S_t, R_t);$$

and for any $t \geq T$, let (C'_t, K'_t, R'_t) be constructed as above, and $(S'_t)_{T \leq t \in \mathbb{N}}$ can be constructed recursively from $(R'_t)_{t \in \mathbb{N}}$ according to the recursive equation $S'_{t+1} = \eta(S'_t - R'_t)$, $\forall t \in \mathbb{N}$.

We see that $(C_t, K_t, S_t, R_t)_{t \in \mathbb{N}}$ is dynamically improved by $(C'_t, K'_t, S'_t, R'_t)_{t \in \mathbb{N}}$. This contradicts the assumption that $(C_t, K_t, S_t, R_t)_{t \in \mathbb{N}}$ is dynamically efficient. And hence, $\lim_{t \to \infty} D_t K_{t+1} = 0$. ▌

Proof of Theorem 3. Suppose the equilibrium allocation $\mathbb{A} = (a_t, b_t, K_t, S_t, R_t)_{t \in \mathbb{N}}$ is not Pareto efficient. Then, by Lemma 4, there is another allocation $\mathbb{A}' = (a'_t, b'_t, K'_t, S'_t, R'_t)_{t \in \mathbb{N}}$ such that

$$U(a'_t, b'_{t+1}) \geq U(a_t, b_{t+1}), \quad \forall t \in \mathbb{N}, \tag{A.3}$$

and

$$b'_0 > b_0.$$

Since $N_{-1}b_0 = (1+r_0)K_0 + p_0S_0$, then we have

$$N_{-1}b_0' > (1+r_0)K_0 + p_0S_0 + \epsilon \tag{A.4}$$

for some $\epsilon > 0$.

Now take $t \in \mathbb{N}$ arbitrarily. For an individual of generation-t, by the definition of equilibrium, we have

$$(a_t, b_{t+1}, (S_t - R_t)) \in \arg\max_{(a,b,X)} U(a,b),$$

subject to

$$a + \frac{b}{1+r_{t+1}} \leq \omega_t + \frac{1}{N_t}\left(\frac{p_{t+1}G(X)}{1+r_{t+1}} - p_tX\right),$$

then

$$(a_t, b_{t+1}) \in \arg\max_{(a,b)} U(a,b),$$

subject to

$$a + \frac{b}{1+r_{t+1}} \leq \omega_t + \frac{1}{N_t}\left(\frac{p_{t+1}G(S_t - R_t)}{1+r_{t+1}} - p_t(S_t - R_t)\right).$$

By (A.3), we have

$$a_t' + \frac{b_{t+1}'}{1+r_{t+1}} \geq \omega_t + \frac{1}{N_t}\left(\frac{p_{t+1}G(S_t - R_t)}{1+r_{t+1}} - p_t(S_t - R_t)\right).$$

In addition, since G is concave and holds the generalized Hotelling rule:

$$\frac{p_{t+1}G'(S_t - R_t)}{1+r_{t+1}} = p_t,$$

then the function $\frac{p_{t+1}G(X)}{1+r_{t+1}} - p_tX$ of X, for given p_t, p_{t+1}, r_{t+1}, takes its maximum at $X = (S_t - R_t)$.

Therefore,

$$\frac{p_{t+1}G(S_t - R_t)}{1+r_{t+1}} - p_t(S_t - R_t) \geq \frac{p_{t+1}G(S_t' - R_t')}{1+r_{t+1}} - p_t(S_t' - R_t').$$

Noticing $S_{t+1}' = G(S_t' - R_t')$, we get

$$a_t' + \frac{b_{t+1}'}{1+r_{t+1}} \geq \omega_t + \frac{1}{N_t}\left(\frac{p_{t+1}S_{t+1}'}{1+r_{t+1}} - p_t(S_t' - R_t')\right), \quad \forall t \in \mathbb{N},$$

and hence,

$$\begin{aligned} & D_t N_t a'_t + D_{t+1} N_t b'_{t+1} \\ \geq \ & D_t \omega_t N_t + \left(D_{t+1} p_{t+1} S'_{t+1} - D_t p_t S'_t\right) + D_t p_t R'_t, \quad \forall t \in \mathbb{N}. \end{aligned} \quad \text{(A.5)}$$

Then, for sufficiently large τ, summing the inequality (A.4) and the inequalities in (A.5) for $t = 0$ through $t = \tau - 1$, yields

$$\begin{aligned} & \sum_{t=0}^{\tau-1} D_t (N_t a'_t + N_{t-1} b'_t) + D_\tau N_{\tau-1} b'_\tau \\ \geq \ & (1 + r_0) K_0 + \sum_{t=0}^{\tau-1} D_t (\omega_t N_t + p_t R'_t) + D_\tau p_\tau S'_\tau + \epsilon. \end{aligned} \quad \text{(A.6)}$$

Noticing the zero maximum profit for any firm and the conditions of feasibility, we have that for any $t \in \mathbb{N}$,

$$(1 + r_t) K'_t + \omega_t N_t + p_t R'_t \geq F^t(K'_t, N_t, R'_t) \geq N_t a'_t + N_{t-1} b'_t + K'_{t+1}.$$

Therefore, for any $t \in \mathbb{N}$,

$$\begin{aligned} & D_{t-1} K'_t + D_t \omega_t N_t + D_t p_t R'_t \\ \geq \ & D_t N_t a'_t + D_t N_{t-1} b'_t + D_t K'_{t+1}. \end{aligned} \quad \text{(A.7)}$$

Summing the inequalities in (A.7) for $t = 0$ through $t = \tau$, yields

$$\begin{aligned} & (1 + r_0) K_0 + \sum_{t=0}^{\tau} D_t \left(\omega_t N_t + p_t R'_t\right) \\ \geq \ & \sum_{t=0}^{\tau} D_t \left(N_t a'_t + N_{t-1} b'_t\right) + D_\tau K'_{\tau+1}. \end{aligned} \quad \text{(A.8)}$$

By summing (A.6) and (A.8), we obtain

$$D_\tau \omega_\tau N_\tau \geq D_\tau (N_\tau a'_\tau + K'_{\tau+1}) + D_\tau p_\tau (S'_\tau - R'_\tau) + \epsilon \geq \epsilon.$$

Then we have

$$\underline{\lim}_{\tau \to \infty} D_\tau \omega_\tau N_\tau > 0.$$

We get a contradiction. Therefore the equilibrium is Pareto efficient. ∎

Proof of Theorem 4. Suppose (4) holds. We construct a Pareto improvement of the equilibrium allocation. To this end, notice that for any $t \in \mathbb{N}$,

$$C_t + K_{t+1} = F(K_t, B_t N_t, R_t),$$

where $C_t = N_t a_t + N_{t-1} b_t$ is the total consumption at time t. Let

$$c_t = \frac{C_t}{B_t N_t}, \quad k_t = \frac{K_t}{B_t N_t}, \quad z_t = \frac{R_t}{B_t N_t}.$$

Then,

$$c_t + \mu k_{t+1} = f(k_t, z_t).$$

In the sequel, fix $(z_t)_{t\in\mathbb{N}}$. Let $x_t = \mu k_t$, $\phi(x, z) = f(x/\mu, z)$. Then,

$$c_t + x_{t+1} = \phi(x_t, z_t), \quad \forall t \in \mathbb{N}.$$

By (4), there exist $\varepsilon \in (0, 1)$ and $\tau \in \mathbb{N}$ such that for any $t \geq \tau$,

$$\frac{x_t \phi_x(x_t, z_t)}{x_{t+1}} < \varepsilon.$$

If we can construct a sequence $(c'_t, x'_t)_{t\in\mathbb{N}}$ such that $c'_\tau > c_\tau$, $c'_t = c_t$, $\forall t \neq \tau$, and $x'_t = x_t$, $\forall t \leq \tau$, and $x'_t > 0$, $\forall t > \tau$, then we can get a Pareto improvement of the equilibrium allocation.

Now, fix $(x_t)_{t\leq\tau}$ and $(c_t)_{t\neq\tau}$. Let $x_{\tau+1}$ decrease a bit, then accordingly, c_τ will increase strictly, and then, for any $t > \tau$, x_{t+1} will decrease as well.

We attempt to prove that there exists a $x'_{\tau+1} \in (0, x_{\tau+1})$ such that when $x_{\tau+1}$ decreases to $x'_{\tau+1}$, then accordingly, for any $t > \tau$, x_{t+1} will decreases to some $x'_{t+1} > 0$.

In fact, first of all, by $\sup_{t\in\mathbb{N}} z_t < \infty$, we know that there exists $Z > 0$ such that $z_t \in [0, Z]$ for any $t \in \mathbb{N}$.

In addition, $\lim_{x\to\infty} \phi_x(x, z) < 1$ uniformly for $z \in [0, Z]$ and either $\liminf_{t\to\infty} x_t > 0$ or $\phi_x(0, z) < \infty$ for any $z \geq 0$. It follows that there exist $0 \leq \underline{x} < \overline{x} < \infty$ such that $(x_t)_{t\in\mathbb{N}}$ is bounded in $[\underline{x}, \overline{x}]$ and $\phi_x(x, z)$ is well defined in $[\underline{x}, \overline{x}] \times [0, Z]$; and $\min_{t\in\mathbb{N}} \phi_x(x_t, z_t) \geq \min_{t\in\mathbb{N}} \phi_x(\overline{x}, z_t) =: m > 0$.

Take $\varepsilon' \in \big(0, m(\varepsilon^{-1} - 1)\big)$. Noticing the uniform continuity of $\phi_x(x, z)$ in $[\underline{x}, \overline{x}] \times [0, Z]$, we have that there exists $\delta \in (0, \overline{x})$ such that for any $x, x' \in [\underline{x}, \overline{x}]$,

$$|\phi_x(x', z_t) - \phi_x(x, z_t)| \leq \varepsilon', \quad \forall t \in \mathbb{N},$$

if only $|x' - x| \leq \delta$.

Now, take $x'_{\tau+1}$ such that

$$0 < \frac{x_{\tau+1} - x'_{\tau+1}}{x_{\tau+1}} < \frac{\delta}{\overline{x}}.$$

Then, for any $t > \tau$,

$$\begin{aligned}
0 &< \frac{x_{t+1} - x'_{t+1}}{x_{t+1}} = \frac{\phi(x_t, z_t) - \phi(x'_t, z_t)}{x_{t+1}} \leq \frac{\phi_x(x'_t, z_t)(x_t - x'_t)}{x_{t+1}} \\
&= \left[\frac{\phi_x(x'_t, z_t) - \phi_x(x_t, z_t)}{\phi_x(x_t, z_t)} + 1\right] \cdot \frac{x_t \phi_x(x_t, z_t)}{x_{t+1}} \cdot \frac{(x_t - x'_t)}{x_t} \\
&\leq \left[\frac{\phi_x(x'_t, z_t) - \phi_x(x_t, z_t)}{\phi_x(\overline{x}, z_t)} + 1\right] \cdot \frac{x_t \phi_x(x_t, z_t)}{x_{t+1}} \cdot \frac{(x_t - x'_t)}{x_t} \\
&\leq \left[\frac{\phi_x(x'_t, z_t) - \phi_x(x_t, z_t)}{m} + 1\right] \cdot \frac{x_t \phi_x(x_t, z_t)}{x_{t+1}} \cdot \frac{(x_t - x'_t)}{x_t} \\
&\leq \frac{x_t - x'_t}{x_t},
\end{aligned}$$

if only

$$0 < \frac{x_t - x'_t}{x_t} < \frac{\delta}{\overline{x}}.$$

It follows that $x'_t > 0$, $\forall t > \tau$. Therefore, such a construction of Pareto improvement of the equilibrium allocation is feasible. ▮

Proof of Theorem 5. For any $t \in \mathbb{N}$, by solving the utility maximization problem for an individual of generation-t, we obtain

$$N_t a_t = \frac{1}{1+\rho} I_t, \quad N_t \frac{b_{t+1}}{1 + r_{t+1}} = \frac{\rho}{1+\rho} I_t.$$

We know that there exist $\varepsilon \in (0,1)$ and $T \in \mathbb{N}$ such that for any $t \geq T$,

$$\frac{1 + r_t}{1 + i_t} < \varepsilon.$$

Now, for any $t \geq T$, consider the function of $\theta \in [0,1]$:

$$f_t(\theta) = \ln\left(a_t(1-\theta)\right) + \rho \ln\left(b_{t+1} + \frac{N_{t+1}}{N_t} a_{t+1}\theta\right).$$

It is easy to see that $f'_t(\theta) > 0$ for any $\theta \in [0, \theta^*_t)$, where

$$\theta^*_t = \frac{\rho}{1+\rho}\left[1 - \frac{1+r_t}{1+i_t}\right].$$

Let

$$\theta^* = \frac{\rho}{1+\rho}(1-\varepsilon).$$

Clearly, for any $t \geq T$,

$$\theta_t^* > \theta^*.$$

And hence for any $t \geq T$,

$$f_t(\theta^*) > f_t(0).$$

Then we can construct an allocation $(a_t', b_t', K_t, S_t, R_t)_{t\in\mathbb{N}}$, a Pareto improvement of the equilibrium allocation $(a_t, b_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$, as follows: for any $t < T$,

$$a_t' = a_t, \quad b_t' = b_t;$$

and for any $t \geq T$,

$$a_t' = a_t(1-\theta^*), \quad b_t' = b_t + \frac{N_t}{N_{t-1}} a_t \theta^*.$$

▮

Proof of Corollary 1. Notice that $V_t \geq p_t R_t$ and Pareto efficiency is stronger than dynamic efficiency. Then, by Theorem 2 and Theorem 3, we get the required result. ▮

Proof of Corollary 2. The generalized Hotelling rule implies that for any $t \in \mathbb{N}$,

$$D_{t+1} p_{t+1} R_{t+1} = D_t p_t R_t \frac{R_{t+1}}{R_t G'(S_t - R_t)}.$$

Then, by (8), we get $\lim_{t\to\infty} D_t p_t R_t = 0$, which, by (9), yields $\varliminf_{t\to\infty} D_t \omega_t N_t = 0$. Thus, by Theorem 3, we obtain the required result. ▮

Proof of Corollary 3. Noticing that for any $t \in \mathbb{N}$,

$$D_t \omega_t N_t = K_0 \frac{\omega_t N_t}{(1+r_t)K_t} \prod_{s=0}^{t-1} \frac{1+j_s}{1+r_s},$$

by (10) and (11), we get $\varliminf_{t\to\infty} D_t \omega_t N_t = 0$. Then, by Theorem 3, we obtain the required result. ▮

Proof of Corollary 4. The case $q < 1$ follows from Corollary 3; the case $q > 1$ follows from Theorem 4. ▮

Proof of Proposition 1. The required result follows easily from the following Lemma 5. ▌

Proof of proposition 2. Since for any $t \in \mathbb{N}$, $K_{t+1} = \alpha\delta Y_t$, then

$$D_{t+1}Y_{t+1} = \frac{1}{\alpha}D_{t+1}(1+r_{t+1})K_{t+1} = \frac{1}{\alpha}D_t K_{t+1} = \delta D_t Y_t,$$

therefore $\lim_{t\to\infty} D_t Y_t = 0$, which yields $\lim_{t\to\infty} D_t\omega_t N_t = \lim_{t\to\infty} \beta D_t Y_t = 0$. By Theorem 3, we obtain the required result. ▌

Proof of Proposition 3. The social planner's problem ($\mathbb{P}$) is

$$\begin{aligned} \max \ & \sum_{t=0}^{\infty} \delta^t \left(\delta \ln a_t + \rho \ln b_t\right), \\ \text{s.t.} \ & K_{t+1} = A_t K_t^{\alpha} N_t^{\beta} R_t^{\gamma} - N_t a_t - N_{t-1} b_t, \quad \forall t \in \mathbb{N}, \\ & S_{t+1} = \eta(S_t - R_t), \quad \forall t \in \mathbb{N}, \end{aligned}$$

and all variables are nonnegative, where K_0, S_0 are given. By transformation

$$\begin{gathered} X_t = \xi^{-t} K_t, \quad H_t = \xi^{-1/\gamma} R_t, \quad Z_t = \xi^{-1/\gamma} S_t, \\ \xi^{-(t+1)} N_t a_t = \frac{\delta}{\delta+\rho} c_t, \quad \xi^{-(t+1)} N_{t-1} b_t = \frac{\rho}{\delta+\rho} c_t, \end{gathered}$$

where $\xi = \left((1+g)(1+n)^{\beta}\right)^{1/(1-\alpha)}$, ($\mathbb{P}$) can be reduced to ($\mathbb{P}'$):

$$\begin{aligned} \max \ & \sum_{t=0}^{\infty} \delta^t \ln c_t, \\ \text{s.t.} \ & X_{t+1} = X_t^{\alpha} H_t^{\gamma} - c_t, \quad \forall t \in \mathbb{N}, \\ & Z_{t+1} = \eta(Z_t - H_t), \quad \forall t \in \mathbb{N}, \end{aligned}$$

and all variables are nonnegative, where X_0, Z_0 are given.

The Bellman equation for ($\mathbb{P}'$) is

$$V(X,Z) = \max_{c,H} \left\{ \ln c + \delta V(X^{\alpha}H^{\gamma} - c, \eta(Z-H)) \right\}.$$

One can verify directly that

$$V(X,Z) = \frac{1}{1-\alpha\delta}\left[\alpha \ln X + \frac{\gamma}{\tau}\ln Z\right] + m, \tag{A.9}$$

with some constant m, satisfies the above Bellman equation, and correspondingly, the unique solution for the optimization problem in the right-hand side of the Bellman equation is

$$c = (1-\alpha\delta)\tau^{\gamma} X^{\alpha} Z^{\gamma}, \quad H = \tau Z, \tag{A.10}$$

which is a stationary Markovian strategy for $(\mathbb{P}')$.

Denote the path of state variables by this strategy as $(X_t, Z_t)_{t\in\mathbb{N}}$, which obviously satisfies the TVCs (transversality conditions):

$$\lim_{t\to\infty} \delta^t V(X_t, Z_t) = \lim_{t\to\infty} \delta^t \left[X_t V_1(X_t, Z_t) + Z_t V_2(X_t, Z_t)\right] = 0.$$

Thus, the above V in (A.9) is the value function of $(\mathbb{P}')$, and the strategy in (A.10) is the unique optimal Markovian strategy for $(\mathbb{P}')$.

Consequently, the unique optimal Markovian strategy for $(\mathbb{P})$ is as follows: for any $t \in \mathbb{N}$,

$$a_t = \frac{\beta}{1+\rho} Y_t/N_t, \quad b_t = \left(\alpha + \frac{\gamma}{\tau}\right) Y_t/N_{t-1}, \quad R_t = \tau S_t,$$

where $Y_t = A_t K_t^{\alpha} N_t^{\beta} R_t^{\gamma}$, and the corresponding dynamics of the state variables are that for any $t \in \mathbb{N}$,

$$K_{t+1} = \alpha\delta Y_t, \quad S_{t+1} = (\eta\delta) S_t.$$

We see that the trajectory $(a_t, b_t, K_t, S_t, R_t)_{t\in\mathbb{N}}$ induced by the strategy in (A.10) is just the equilibrium allocation. ▌

Proof of Proposition 5. Denote the steady state of the dynamical system $\mathscr{D}$ as (k, s, z), and denote the limit wage and the limit interest rate of the corresponding equilibrium as ω and r, respectively. We have

$$k = \frac{1}{1+n} \left(\alpha k^{\sigma} + \beta + \gamma z^{\sigma}\right)^{(1-\sigma)/\sigma} \left[\theta - \frac{\gamma(s-z)}{z^{1-\sigma}}\right],$$

$$\alpha k^{\sigma} + \beta + \gamma z^{\sigma} = \frac{1}{1+n} \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \left[\theta - \frac{\gamma(s-z)}{z^{1-\sigma}}\right].$$

It follows that

$$\omega = \beta \left(\alpha k^{\sigma} + \beta + \gamma z^{\sigma}\right)^{(1-\sigma)/\sigma} > 0,$$

$$1 + r = \alpha k^{\sigma-1} \left(\alpha k^{\sigma} + \beta + \gamma z^{\sigma}\right)^{(1-\sigma)/\sigma} = \eta > 1+n,$$

which implies $\lim_{t\to\infty} D_t\omega_t N_t = 0$. Then, by Theorem 3, this equilibrium is Pareto efficient. ▌

Proof of Proposition 6. First of all, no type III equilibrium exists. In fact, otherwise, by (15), for large t, approximately, $z_{t+1} = mz_t^{1-\sigma}$ with some constant $m > 0$, which yields z_t must be bounded from above. This is a contradiction.

If $\eta > 1 + n$, there dos not exist a type I equilibrium with harvesting speed indictor below θ. In fact, otherwise, for large t, approximately, $z_{t+1} = (\eta/(1+n))^{1/(1-\sigma)} z_t$. Then $z_t \not\to 0$ as $t \to \infty$. We get a contradiction. On the other hand, it is easy to verify that there exists really a unique type I equilibrium, and its harvesting speed indictor is θ, and correspondingly, its limit capital per capita is $\pi^{-1}(\theta)$.

Now, we suppose $\eta \leq 1 + n$. Under this assumption, any type I equilibrium corresponds to a $\epsilon \in [0, \theta]$ such that

$$\lim_{t\to\infty} s_t z_t^{\sigma-1} = (\theta - \epsilon)/\gamma, \quad \lim_{t\to\infty} k_t = k,$$

where $k = \pi^{-1}(\epsilon)$. In addition, for large t, approximately, by (15) and (17),

$$z_{t+1} = \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{k}{(\alpha k^\sigma + \beta)^{1/\sigma}} z_t.$$

Therefore,

$$\left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{k}{(\alpha k^\sigma + \beta)^{1/\sigma}} \leq 1,$$

or, equivalently,

$$k^\sigma \left(\eta^{\sigma/(1-\sigma)} - \alpha^{1/(1-\sigma)}\right) \leq \beta\alpha^{\sigma/(1-\sigma)}.$$

Otherwise, $z_t \not\to 0$ as $t \to \infty$. This is a contradiction. Therefore, we define

$$\overline{\theta} = \sup\left\{\epsilon \in [0, \theta] \,\middle|\, (\pi^{-1}(\epsilon))^\sigma \left(\eta^{\sigma/(1-\sigma)} - \alpha^{1/(1-\sigma)}\right) \leq \beta\alpha^{\sigma/(1-\sigma)}\right\},$$

and define $\overline{k} = \pi^{-1}(\overline{\theta})$.

Thus, any type I equilibrium corresponds to a $k \in [0, \overline{k}]$ and a $\epsilon \in [0, \overline{\theta}]$ with $\epsilon = \pi(k)$ such that along this equilibrium path, the limit capital per capita is k, and the harvesting speed indicator is ϵ. ▌

Proof of Proposition 7. Suppose $\eta > 1 + n$. If (19) holds, then by Proposition 5, the unique type II equilibrium is Pareto efficient. If (19) does not hold, then $\nu \leq \frac{1+n}{\eta} < 1$, which implies $\alpha \geq \eta^\sigma \left(2 + \rho^{-1}\right)^{\sigma-1}$. Thus, for the unique type I equilibrium, the limit capital per capita is $k = \pi^{-1}(\theta)$, the limit wage exists and is positive, while the limit interest rate r satisfies

$$\frac{1+r}{1+n} = \frac{\alpha}{\theta} k^{\sigma} = x,$$

and x satisfies

$$(1+n)^{\sigma} x = \alpha \left(x + \frac{\beta}{\theta} \right)^{1-\sigma}.$$

From $\eta > 1+n$, $\alpha \geq \eta^{\sigma} \left(2 + \rho^{-1} \right)^{\sigma - 1}$, we obtain

$$(1+n)^{\sigma} < \alpha \left(1 + \frac{\beta}{\theta} \right)^{1-\sigma}.$$

Therefore, $x > 1$. Then, by Corollary 4, this equilibrium is Pareto efficient.

Suppose $\eta \leq 1+n$. Taking $k \in (0, \overline{k}]$ and $\epsilon \in (0, \overline{\theta}]$ with $\epsilon = \pi(k)$ arbitrarily. The type I equilibrium with limit capital per capita k satisfies the assumption **A3**. Denote its limit interest rate as r, and let

$$\phi = \frac{1+r}{1+n}.$$

From

$$1 + r = \alpha k^{\sigma - 1} \left(\alpha k^{\sigma} + \beta \right)^{(1-\sigma)/\sigma},$$

we obtain

$$(1+n)^{\sigma} \phi = \alpha \left(\phi + \frac{\beta}{\epsilon} \right)^{1-\sigma}.$$

It is easy to see that there exists a $\underline{\theta} \in (0, \overline{\theta}]$ such that $\phi > 1$, if $\epsilon < \underline{\theta}$; $\phi < 1$, if $\epsilon > \underline{\theta}$. Denote $\underline{k} = \pi^{-1}(\underline{\theta})$. Then, by Corollary 4, this equilibrium is Pareto efficient, if $k < \underline{k}$; it is Pareto inefficient, if $k > \underline{k}$. ▌

Proof of Proposition 8. (i) Suppose $\eta \leq 1+n$. It is easy to see that there is no equilibrium of type III, but there is a continuum of equilibria of type I: for any $\epsilon \in [0, \theta]$, there is an equilibrium of type I such that as $t \to \infty$,

$$\frac{s_t - z_t}{z_t^{1-\sigma}} \to (\theta - \epsilon)/\gamma, \quad k_t \to 0.$$

(ii) Suppose $\eta > 1+n$. First of all, there is no type III equilibrium with harvesting speed indictor 0. In fact, otherwise, by (16), we have that for large t, $z_{t+1} \leq z_t/2$, which contradicts $z_t \to \infty$ as $t \to \infty$.

If there is a type III equilibrium with harvesting speed indictor $\epsilon \in (0, \theta]$ and the limit capital per capita $k > 0$, then as $t \to \infty$,

$$\frac{s_t - z_t}{z_t^{1-\sigma}} \to \frac{\theta - \epsilon}{\gamma}, \quad k_t \to k,$$

and

$$k = \frac{\epsilon}{1+n} \left(\alpha k^\sigma + \beta\right)^{(1-\sigma)/\sigma}.$$

For simplicity, let $x = \frac{\alpha}{\epsilon} k^\sigma$. Then,

$$(1+n)^\sigma x = \alpha \left(x + \frac{\beta}{\epsilon}\right)^{1-\sigma}. \tag{A.11}$$

In addition, by (17), for large t, approximately,

$$z_{t+1} = \frac{\epsilon}{1+n} \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{z_t}{\alpha k^\sigma + \beta},$$

then it must hold that

$$1 \leq \frac{\epsilon}{1+n} \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{1}{\alpha k^\sigma + \beta} = \left(\frac{\eta}{\alpha}\right)^{1/(1-\sigma)} \frac{1}{(1+n)\left[x + \frac{\beta}{\epsilon}\right]},$$

or, equivalently,

$$\frac{\eta}{\alpha} \geq (1+n)^{1-\sigma} \left[x + \frac{\beta}{\epsilon}\right]^{1-\sigma} = (1+n)\frac{x}{\alpha},$$

that is,

$$x \leq \frac{\eta}{1+n}. \tag{A.12}$$

To sum up, there is a type III equilibrium with harvesting speed indicator $\epsilon \in (0, \theta]$, if and only if (38) has a solution satisfying (39).

For given ϵ, the equation (38) for x has a solution, if and only if $\epsilon \geq \epsilon_*$, which implies $\theta \geq \epsilon_*$, or, equivalently, $\alpha \leq \alpha^*$. In addition, since (19) is not satisfied, then

$$\frac{\eta}{1+n} \left[\left(\frac{\eta^\sigma}{\alpha}\right)^{1/(1-\sigma)} - 1\right] \leq \frac{\beta}{\theta}.$$

With the above observations, the remainder of the results can be proven. ∎

Proof of Proposition 9. First, consider an equilibrium of type I. Since for any $t \in \mathbb{N}$,

$$1 + r_t = \alpha k_t^{\sigma-1} \left(\alpha k_t^\sigma + \beta + \gamma z_t^\sigma\right)^{(1-\sigma)/\sigma},$$

$$\omega_t = \beta \left(\alpha k_t^\sigma + \beta + \gamma z_t^\sigma\right)^{(1-\sigma)/\sigma},$$

then,

$$\begin{aligned} \frac{1}{1+r_{t+1}} \frac{\omega_{t+1} N_{t+1}}{\omega_t N_t} &= \frac{1+n}{\alpha} \frac{k_{t+1}^{1-\sigma}}{\left(\alpha k_t^\sigma + \beta + \gamma z_t^\sigma\right)^{(1-\sigma)/\sigma}} \\ = \frac{k_{t+1}^{-\sigma}}{\alpha} \left[\theta - \frac{\gamma(s_t - z_t)}{z_t^{1-\sigma}}\right] &\le \frac{\theta}{\alpha} k_{t+1}^{-\sigma} \to 0, \quad \text{as} \quad t \to \infty, \end{aligned}$$

which implies $\lim_{t\to\infty} D_t \omega_t N_t = 0$. Then, by Theorem 3, the equilibrium is Pareto efficient.

Next, for any type III equilibrium with the harvesting speed indicator $\epsilon \in [\epsilon^*, \theta]$ and limit capital per capita k and limit interest rate r, we have

$$\frac{1+r}{1+n} = x = \frac{\alpha}{\epsilon} k^\sigma,$$

where x satisfies

$$(1+n)^\sigma x = \alpha \left(x + \frac{\beta}{\epsilon}\right)^{1-\sigma}.$$

According to Corollary 4, this equilibrium is Parto-efficient if $x > 1$ and Pareto inefficient if $x < 1$. Therefore, by determining whether $x > 1$ or $x < 1$, one can prove the remainder of the result. ∎

Proof of Proposition 11. It is easy to verify that if $\beta < \kappa$, then $x^* \in \left(0, \frac{B}{2}\left(1 - \frac{1}{\lambda}\right)\right)$, which implies $G'(x^*) > 1$; if $\beta > \kappa$, then $x^* \in \left(\frac{B}{2}\left(1 - \frac{1}{\lambda}\right), \frac{B}{2}\right)$, which implies $G'(x^*) < 1$.

For any $t \in \mathbb{N}$, denote $x_t = S_t - R_t$. Noticing that as $t \to \infty$,

$$x_t \to x^*, \quad R_t \to R^*, \quad S_t \to S^*,$$

and for any $t \in \mathbb{N}$,

$$\begin{aligned} I_t &= \omega_t N_t + \left[\frac{p_{t+1} G(x_t)}{1 + r_{t+1}} - p_t x_t\right] \\ &= \omega_t N_t + p_t \left[\frac{G(x_t)}{G'(x_t)} - x_t\right] \\ &= Y_t \left\{\beta + \frac{\gamma}{R_t} \left[\frac{G(x_t)}{G'(x_t)} - x_t\right]\right\}, \end{aligned}$$

by (23) and (25), we have

$$\lim_{t\to\infty}\frac{1+i_t}{1+r_t}=\lim_{t\to\infty}\frac{I_t}{(1+r_t)I_{t-1}}=\lim_{t\to\infty}\frac{Y_t}{(1+r_t)Y_{t-1}}$$
$$=\lim_{t\to\infty}\frac{K_{t+1}}{(1+r_t)K_t}=\lim_{t\to\infty}\frac{K_{t+1}}{\alpha Y_t}=\lim_{t\to\infty}\frac{R_{t+1}}{R_tG'(x_t)}=\frac{1}{G'(x^*)}.$$

If $\beta>\kappa$, then

$$\lim_{t\to\infty}\frac{1+i_t}{1+r_t}>1,$$

thus, by Theorem 5, the equilibrium is Pareto inefficient.

If $\beta<\kappa$, then

$$\lim_{t\to\infty}\frac{Y_t}{(1+r_t)Y_{t-1}}<1,$$

therefore,

$$\lim_{t\to\infty}D_t\omega_tN_t=\beta\lim_{t\to\infty}D_tY_t=0,$$

thus, by Theorem 3, the equilibrium is Pareto efficient. ▮

REFERENCES

Abel, A. B., N. G. Mankiw, L. H. Summers, and R. J. Zeckhauser , 1989. Assessing Dynamic Efficiency: Theory and Evidence. *The Review of Economic Studies* **56(1)**, 1-20.

Agnani, B., M. Gutierrez, and A. Iza, 2005. Growth in overlapping generation economies with non-renewable resources. *Journal of Environmental Economics and Management* **50**, 387-407.

Balasko, Y., and K. Shell, 1980. The overlapping generations Model, I: The Case of Pure Exchange without Money. *Journal of Economic Theory* **23**, 281-306.

Becker, R. A., and T. Mitra, 2012. Efficient Ramsey equilibria. *Macroeconomic Dynamics* **16**, 18-32.

Benveniste, L. M., and D. Gale, 1975. An Extension of Cass Characterization of Infinite Efficient Production Programs. *Journal of Economic Theory* **10**, 229-238.

Cass, D., 1972. On capital overaccumulation in aggregative, neoclassical model of Economic growth: A complete characterization. *Journal of Economic Theory* **4**, 200-223.

Chattopadhyay, J. S., 2008. The Cass criterion, the net dividend criterion, and optimality. *Journal of Economic Theory* **139**, 335-352.

Diamond, P., 1965. National debt in a neoclassical growth model. *American Economic Review* **55**, 1126-1150.

Ding, N., and B. C. Field, 2005. Natural Resource Abundance and Economic Growth. *Land Economics* **81(4)**, 496-502.

Farmer, K., 2000. Intergenerational natural-capital equality in an overlapping generations model with logistic regeneration. *Journal of Economics* **72(2)**, 129-152.

Farmer, K., and B. Bednare-Friedl, 2010. Intertemporal Resource Economics. An Introduction to the Overlapping Generations Approach. Berlin Heidelberg: Springer-Verlag.

Farmer, K., and B. Bednare-Friedl, 2017. Existence and efficiency of stationary states in a renewable resource based OLG model with different harvest costs. *Studia Universitatis Babes-Bolyai Oeconomiica* **62(3)**, 3-32.

Geanakoplos, J., and H. Polemarchakis, 1991. Overlapping Generations. Chapter 35 in Handbook of Mathematical Economics, Volume IV, Edited by Werner Hildenbrand and Hugo Sonnenschein. North Holland: Elsevier Science Publishers, 1899-1962.

Hellwig, M. F., 2024. Dynamic efficiency and inefficiency in a class of overlapping generations economies with multiple assets. Max Planck Institute For Research on Collective Goods. Discussion Paper 2024/8.

Homburg, S., 1992. Efficient Economic Growth (Microeconomic Studies). Berlin,Heidelberg: Springer-Verlag.

Hotelling, H., 1931. The economics of exhaustible resources. *Journal of Political Economy* **39**, 137-175.

Kemp, M. C., and N. V. Long, 1979. The under-exploitation of natural resources: A model with overlapping generations. *Economic Record* **55**, 214-221.

Koskela, E., M. Ollikainen, and M. Puhakka, 2002. Renewable resources in an overlapping generations economy without capital. *Journal of Environmental Economics and Management* **43(3)**, 497-517.

Krautkraemer, J. A., and R. G. Batina, 1999. On sustainability and intergenerational transfers with a renewable resource. *Land Economics* **75(2)**, 167-184.

Malinvaud, E., 1953. Capital accumulation and efficient allocation of resources. *Econometrica* **21(2)**, 233-268.

Mehlum, H., K. Moene, and R. Torvik, 2006, Institutions and the resource curse. *Economic Journal* **116(508)**,1–20.

Miao, J., 2020. Economic Dynamics in Discrete Time. Cambridge, Massachusetts: The MIT Press.

Mitra, T., 1978, Efficient growth with exhaustible resources in a neoclassical model. *Journal of Economic Theory* **17**, 114-129.

Mourmouras, A., 1991. Competitive equilibria and sustainable growth in a life-cycle model with natural resources. *Scandinavian Journal of Economics* **93(4)**, 585-591.

Olson, L. J., and K. C. Knapp, 1997. Exhaustible resource allocation in an overlapping generations economy. *Journal of Environmental Economics and Management* **32**, 277-292.

Rhee, C., 1991. Dynamic inefficiency in an economy with land. *Review of Economic Studies* **58(4)**, 791-797.

Sachs, J. D., and A. M. Warner, 1995. Natural Resource Abundance and Economic growth. *NBER working paper* 5398.

Sachs, J. D., and A. M. Warner, 1999. The big push, natural resource booms and growth. *Journal of Development Economics* **59**, 43–76.

Sachs, J. D., and A. M. Warner, 2025. Natural Resource Abundance and Economic growth. *Annals of Economics and Finance* **26(2)**, 401-440.

Tirole, J., 1985. Asset bubbles and overlapping generations. *Econometrica* **53**, 1071-1100.

Wick, K., and E. Bulte, 2009. The Curse of Natural Resources. *Annual Review of Resource Economics* 1, 139–156.

Wilson, C. A., 1981. Equilibrium in dynamic models with an infinity of agents. *Journal of Economic Theory* 24, 95-111.